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Elder, Philip Robert.

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Philip Robert Elder

NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

A THEORETICAL ANALYSIS OF UNSTEADY
TRANSONIC CASCADE FLOW

by

Philip Robert Elder

Thesis Advisor: M.F. Platzer

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by

Philip Robert Elder
Lieutenant Commander, United States Navy
B.S., United States Naval Academy, 1965

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ABSTRACT

This thesis presents an analysis of transonic potential flow through an oscillating unstaggered thin plate cascade. A collocation technique is used involving the superposition of adjacent blade "isolated foil" potentials with interference potentials of unknown strength. Imposition of flow tangency requirements leads to integral equations for the unknown source distributions of the interference potentials. Results presented include the interference coefficients for the steady and oscillating cases.

The steady case is extended to the determination of the potentials along the blade and compares favorably with a parallel solution using a Laplace transform approach to the sonic wind tunnel wall interference problem.

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			Section
A	=	$K^2/(\lambda^2+4K^2)$	III,A,B
a	=	velocity of sound	II
AL	=	upper limit of the correction integral	A
B	=	$\lambda R^2/4$	III,A,B
b	=	blade half chord	III
C	=	reference length, blade chord	II
C	=	Fresnel cosine integral	A
COS	=	cosine	III,A
C_p	=	pressure coefficient	II
D	=	$2K^3/(\lambda^2+4K^2)$	III,A,B
d	=	wall to blade distance	III,B
E	=	$KR^2/2$	III,A,B
e	=	energy of an individual particle	II
EXP	=	exponential function	III,A,B
erf	=	error function	B
h	=	function describing the position of a blade chord line	II,III
i	=	$\sqrt{-1}$	II,III,A
I_B	=	integral (A-III-1)	III,A
I_N	=	integral (III-22)	III
I_{NI}	=	integral (III-27)	III,A
I_{NR}	=	integral (III-26)	III,A
K	=	reduced frequency, $K = \omega C/U_\infty$	II,III,A,B
L	=	the number of points on the blade at which the boundary conditions are enforced	III,A,B

$\mathcal{L}(A)$	=	Laplace transform of A	III
N	=	summation index	III,A,B
M	=	Mach number	II
P	=	blade spacing	III
p	=	pressure	II
Q	=	$\lambda + 2iK$	II,III,A
R	=	gas constant	II
R	=	nondimensional blade spacing	III,A,B
$\mathcal{S}$	=	Fresnel sine integral	A
s	=	entropy of the individual particle	II
SIN	=	sine	III,A
T	=	absolute temperature	II
t	=	time	II
U	=	X velocity component	II,III,A,B
u	=	perturbation X velocity	II
V	=	Y velocity component	II
v	=	perturbation Y velocity	II,III,A,B
W	=	absolute velocity	II
X	=	abscissa coordinate	II
X	=	nondimensional abscissa coordinate	III,A,B
Y	=	ordinate coordinate	II
Y	=	nondimensional ordinate coordinate	III,A,B
α	=	maximum oscillation angle measured from the abscissa	III
γ	=	ratio of specific heats	II
$\mathcal{D}$	=	interference function coefficient	III,A,B
ε	=	a small quantity of perturbation magnitude	II

η	=	ordinate coordinate	III
θ_m	=	normal velocity function coefficient	III,A
λ	=	thickness parameter, $\lambda = (\gamma+1)\phi_{1XX}$	III,A,B
ξ	=	abscissa coordinate	III
ρ	=	density	II
Φ,Ψ	=	velocity potential	II
ϕ	=	perturbation potential	II
ϕ_1	=	perturbation potential, steady component	II
Φ_m^o	=	isolated blade perturbation potential	III
Φ_m	=	interference perturbation potential	III
ψ	=	perturbation potential, unsteady component	III,A,B
ω	=	circular frequency	
$\frac{dA}{dB}$	=	derivative of A with respect to B	II
$\frac{\partial A}{\partial B}, A_B$	=	partial of A with respect to B	II,III,A,B
$\frac{DA}{DB}$	=	substantial derivative of A with respect to B	II

SUBSCRIPTS

∞	=	quantity evaluated upstream in steady unperturbed conditions	II,III
x,y,t	=	partial with respect to	II,III,A,B
u	=	upper	II
L	=	lower	II
1	=	steady state quantity	II

o	=	reference blade	III,A,B
1	=	adjacent blade	III

SUPERSCRIPT

*	=	critical condition, M = 1	II
+	=	approaching from the positive side	II
-	=	approaching from the negative side	II

MISCELLANEOUS

$\overline{A}$	=	overbar, nondimensional quantity	II
$\overline{A}$	=	overbar, Laplace transform	III
$\overline{A}$	=	overbar, matrix	A
$\hat{A}$	=	hat, unsteady component	II

COMPUTER VARIABLES

COS	=	cosine
DK	=	reduced frequency, K
DL	=	λ
DLAM	=	λ
EPS	=	convergence criterion
ERFC(A)	=	1 - erf(A)
EXP(A)	=	exponential function, e^A
PI	=	π
RHO	=	R
SIN	=	sine
XA	=	lower limit of an integral
XO	=	upper limit of an integral

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I. INTRODUCTION

Considerable advances are still being made in the development of aircraft turbopropulsion systems. In the quest for higher thrust to weight ratios, one must look to higher stage loading as well as significantly increased turbine inlet temperatures. From the compressor and fan design point of view, this translates primarily into increased tip speeds. Moving toward better blade design, the investigation of transonic and supersonic relative flows is becoming more important not only for increased performance but also for flutter prediction. As shown by Pratt and Whitney, various flutter problems have already been encountered [Fig. 1 as extracted from Ref. 1] and more such problems can be expected in the future.

Recognizing that the actual three dimensional flow fields encountered in a transonic stage are complicated by the effects of rotation, finite blade thicknesses and cambers, boundary layers, and spanwise Mach distributions, it is excessively difficult to model these effects simultaneously. One can hope that eventually a complete analysis will become practical, but at this time the problems must be faced piecemeal. One simplified technique historically used in modeling such flows is the two dimensional cascade.

The following investigation was concerned with transonic cascade flow. This problem had previously been considered

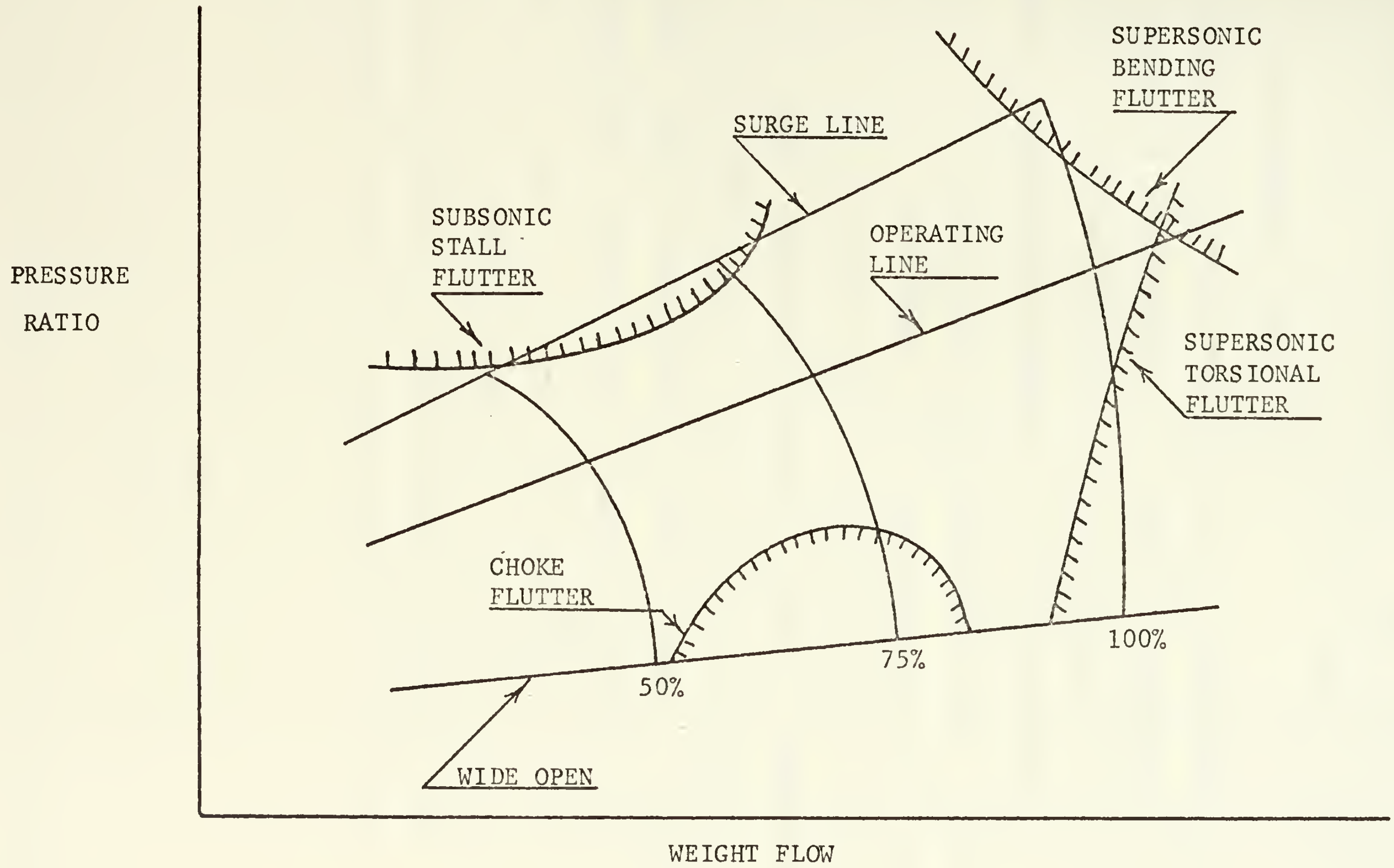


Figure 1. SCHEMATIC COMPRESSOR MAP SHOWING FLUTTER BOUNDARIES FOR FOUR TYPES OF SIMPLE MODE FLUTTER

by Hamamoto [Ref. 2] and Gorelov [Ref. 3]. Also of importance is the related problem of transonic wind tunnel interference. Here the studies of Sandeman [Ref. 4] and Murasaki [Ref. 5] must be mentioned. For a more general exposition of this problem, reference is made to Berndt [Ref. 6] and Goethert [Ref. 7].

Hamamoto based his analysis on the linearized oscillatory transonic flow equations. Gorelov attempted to include the effect of blade thickness by approximating the nonlinear transonic perturbation equation. This approximation, however, is valid only for low supersonic flows. In the present investigation, a transonic flow approximation was introduced based on previous work by Oswatitch [Ref. 8] and Teipel [Ref. 9]. The technique permitted a general formulation for both the steady choke flow problem (in conjunction with Spreiter's local linearization technique [Ref. 10]) and the oscillatory problem. The resulting boundary value problem was solved in a manner similar to that used by Gorelov. The problem formulation, solution technique, numerical evaluation of the recurring integrals, and representative results are described in the following sections.

II. UNSTEADY TRANSONIC FLOW THEORY

A. DEVELOPMENT OF THE GOVERNING TRANSONIC FLOW EQUATIONS

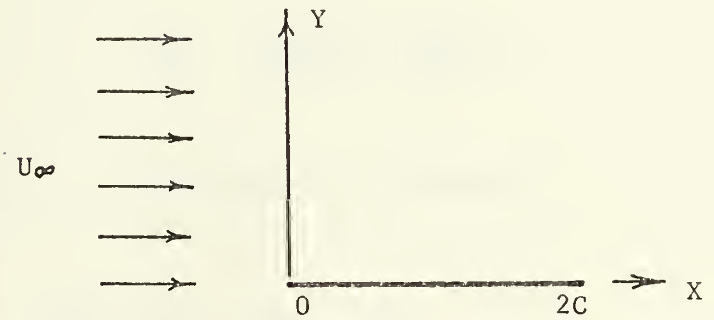


Figure 2. COORDINATE SYSTEM

Consider planar flow of an inviscid adiabatic perfect gas along the positive X axis which is perturbed by a thin body (or thin bodies) executing small transverse unsteady motions. The perfect gas equation is applicable

$$p = \rho RT \tag{II-1}$$

where:

p = pressure

ρ = density

T = absolute temperature

R = gas constant

Furthermore the conservation conditions for mass, momentum, and energy must hold. Denoting the velocity components in the X and Y direction respectively as U and V, the conservation of mass may be expressed as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (U\rho)}{\partial X} + \frac{\partial (VP)}{\partial Y} = 0 \qquad \text{(II-2)}$$

Conservation of momentum is expressed

$$\frac{DU}{Dt} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial X}$$

$$\text{(II-3)}$$

$$\frac{DV}{Dt} = \frac{\partial V}{\partial t} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial Y}$$

Conservation of energy is

$$\frac{D}{Dt}[e + \frac{W^2}{2}] = -\frac{1}{\rho}[\frac{\partial}{\partial X}(UP) + \frac{\partial}{\partial Y}(VP)] \qquad \text{(II-4)}$$

or

$$\frac{Ds}{Dt} = 0 \qquad \text{(II-5)}$$

where

$\frac{D}{Dt}$ = substantial derivative

e = energy for the individual particle

s = entropy for the individual particle

$W^2 = U^2 + V^2$

These equations, of course, exclude finite discontinuities (shocks) in the flow field.

As is well known, an important role is played by certain functions which reduce the number of dependent variables. Most often the velocity potential is introduced. Its existence depends on the condition of irrotationality of the flow. For transonic flows over sufficiently thin bodies (small perturbation assumption!) it is usually possible to work with the velocity potential since any vorticity introduced by occurring shocks is of at least third order in the perturbation velocities. It follows that the velocity can be expressed as

$$U = \frac{\partial \Phi}{\partial X} = \Phi_x \qquad \text{(II-6)}$$

$$V = \frac{\partial \Phi}{\partial Y} = \Phi_y \ .$$

The continuity equation (II-2) may be written

$$-\frac{1}{\rho}\frac{D\rho}{Dt} = \Phi_{xx} + \Phi_{yy}. \qquad \text{(II-7)}$$

Recall that the velocity of sound is

$$a^2 = \left(\frac{dP}{d\rho}\right)_{s=\text{constant}} \qquad \text{(II-8)}$$

and that

$$\frac{D\rho}{Dt} = \frac{d\rho}{dP} \cdot \frac{DP}{Dt} = \frac{1}{a^2} \frac{DP}{Dt}. \qquad \text{(II-9)}$$

Therefore the continuity equation (II-7) becomes

$$\Phi_{xx} + \Phi_{yy} + \frac{1}{a^2\rho}(\Phi_x P_x + \Phi_y P_y + P_t) = 0 \; . \qquad \text{(II-10)}$$

Integration of the Euler equations along a streamline gives the unsteady Bernoulli equation

$$\Phi_t + \frac{W^2}{2} + \int \frac{dP}{\rho} = F(t) \qquad \text{(II-11)}$$

where $F(t) = \frac{1}{2}U_\infty^2$ for uniform parallel flow at infinity. Taking the partial of (II-11) with respect to t, one has

$$\frac{1}{\rho} P_t = -\frac{\partial}{\partial t}[\Phi_t + \frac{W^2}{2}] \; . \qquad \text{(II-12)}$$

Inserting (II-3) and (II-12) for P_x, P_y, and P_t in (II-10) and utilizing (II-6) one obtains the following partial differential equation in Φ.

$$(1-\frac{\Phi_x^2}{a^2})\Phi_{xx} + (1-\frac{\Phi_y^2}{a^2})\Phi_{yy} - \frac{2\Phi_x\Phi_y}{a^2} \Phi_{xy}$$

$$- \frac{2\Phi_x}{a^2} \Phi_{xt} - \frac{2\phi_y}{a^2} \Phi_{yt} - \frac{1}{a^2} \Phi_{tt} = 0 \qquad \text{(II-13)}$$

The usual way to simplify this equation is to neglect all product terms obtaining

$$(1-M_\infty^2)\Phi_{xx} + \Phi_{yy} - 2\frac{M_\infty}{a_\infty}\Phi_{xt} - \frac{1}{a_\infty^2}\Phi_{tt} = 0 \qquad \text{(II-14)}$$

which is the well known unsteady linearized potential equation valid for the supersonic and subsonic regimes. However, for the case near $M_\infty = 1$, such a "naive" linearization cannot, in general, be carried out. Consider the following analysis for the steady case.

Define the perturbation potential, ϕ, such that

$$\Phi = U_\infty X + \phi. \qquad \text{(II-15)}$$

For the steady case, the Bernoulli equation (II-11) is

$$\frac{W^2}{2} + \int\frac{dP}{\rho} = \text{constant}\ . \qquad \text{(II-16)}$$

Now since

$$\int\frac{dP}{\rho} = C_p\ T = \frac{\gamma}{\gamma-1} R\ T$$

and

$$a^2 = \left(\frac{\partial P}{\partial \rho}\right)_s = \gamma R\ T$$

Equation (II-16) becomes

$$\frac{W^2}{2} + \frac{a^2}{\gamma-1} = \text{constant} \tag{II-17}$$

The constant is evaluated by noting that at the critical Mach number

$$W^* = a^*$$

and (II-17) becomes

$$a^2 = \frac{\gamma+1}{2} a^{*2} - \frac{\gamma-1}{2} W^2 \tag{II-18}$$

With (II-18) and (II-15) in (II-13), one has

$$\begin{aligned} \phi_{xx}(a_\infty^2 - U_\infty^2) + \phi_{yy} a_\infty^2 &= \phi_{xx}[(\gamma+1)U_\infty \phi_x + \frac{\gamma+1}{2} \phi_x{}^2 \\ &+ \frac{\gamma-1}{2} \phi_y{}^2] + 2\phi_{xy}[U_\infty \phi_y + \phi_x \phi_y] \\ &+ \phi_{yy}[(\gamma-1)U_\infty \phi_x + \frac{\gamma-1}{2} \phi_x{}^2 + \frac{\gamma+1}{2} \phi_y{}^2] \end{aligned} \tag{II-19}$$

As was mentioned previously, under the Prandtl assumption, one would neglect the product terms on the right-hand side (RHS). However, for $M_\infty \cong 1$, a more careful analysis is in order. To this end one introduces the Prandtl-Glauert transformation:

$$\phi = \varepsilon\bar{\phi}(\bar{X},\bar{Y}) \tag{II-20}$$

where $\bar{X} = X$

$$\bar{Y} = Y\sqrt{1-M_\infty^2} \tag{II-21}$$

$$M_\infty = \frac{U_\infty}{a_\infty} \tag{II-22}$$

Here, one requires $\bar{\phi}$ to be of order unity, therefore ε assumes the order of magnitude of ϕ.

Equation (II-18) becomes

$$\begin{aligned}
\varepsilon\bar{\phi}_{\bar{x}\bar{x}}(1-M_\infty^2)a_\infty^2 + \varepsilon\bar{\phi}_{\bar{y}\bar{y}}(1-M_\infty^2)a_\infty^2 &= \varepsilon^2(\gamma+1)U_\infty\bar{\phi}_{\bar{x}\bar{x}}\bar{\phi}_{\bar{x}} \\
&+ \varepsilon^3\frac{\gamma+1}{2}\bar{\phi}_{\bar{x}\bar{x}}\bar{\phi}_{\bar{x}}^2 + \varepsilon^3\frac{\gamma-1}{2}(1-M_\infty^2)\bar{\phi}_{\bar{x}\bar{x}}\bar{\phi}_{\bar{y}}^2 \\
&+ 2\varepsilon^2\bar{\phi}_{\bar{x}\bar{y}}(1-M_\infty^2)U_\infty\bar{\phi}_{\bar{y}} + 2\varepsilon^3\bar{\phi}_{\bar{x}\bar{y}}\bar{\phi}_{\bar{x}}\bar{\phi}_{\bar{y}}(1-M_\infty^2) \\
&+ \varepsilon^2(1-M_\infty^2)(\gamma-1)U_\infty\bar{\phi}_{\bar{x}}\bar{\phi}_{\bar{y}\bar{y}} + \varepsilon^3\frac{\gamma+1}{2}(1-M_\infty^2)^2\bar{\phi}_{\bar{y}\bar{y}}\bar{\phi}_{\bar{y}}^2 \\
&+ \varepsilon^3\frac{(\gamma-1)}{2}(1-M_\infty^2)\bar{\phi}_{\bar{y}\bar{y}}\bar{\phi}_{\bar{x}}^2
\end{aligned} \tag{II-23}$$

Now the terms on the RHS may be neglected for suitably small ε. To investigate the validity of such a measure, one may compare the largest term on the RHS with a term from the LHS. This term is the first term of order ε^2, the remaining terms being of order ε^3 or $\varepsilon^2(1-M_\infty^2)$. Consider

the ratio

$$E = \frac{U_\infty \varepsilon(\gamma+1)\, \bar{\phi}_{\bar{x}}}{a_\infty^2\, (1-M_\infty^2)} \qquad \text{(II-24)}$$

It is clear that the nonlinear term becomes of higher order for the subsonic and supersonic cases but must be retained for transonic flows. This consideration can be generalized to unsteady transonic flows thus leading to the transonic small perturbation equation as the governing equation:

$$\left[1-M_\infty^2-(\gamma+1)\frac{M_\infty^2}{U_\infty}\phi_x\right]\phi_{xx} + \phi_{yy} - 2\frac{M_\infty}{a_\infty}\phi_{xt} - \frac{1}{a_\infty^2}\phi_{tt} = 0 \qquad \text{(II-25)}$$

A rigorous justification for this equation was first given by Lin, Reissner, and Tsien [Ref. 11] and by Landahl [Ref. 12].

A further simplification of this equation is possible if the transonic flow is sufficiently unsteady. Although this can be shown by mathematical order of magnitude considerations in these same references, it is instructive to consider the physical aspects of the transonic wave propagation problem. Figure 3 shows an acoustic source in transonic flow. It may be seen that the parts of the wave fronts moving upstream, referred to as receding waves, move away very slowly from the source. The downstream or advancing waves move with a velocity approximately equal to twice the speed of sound.

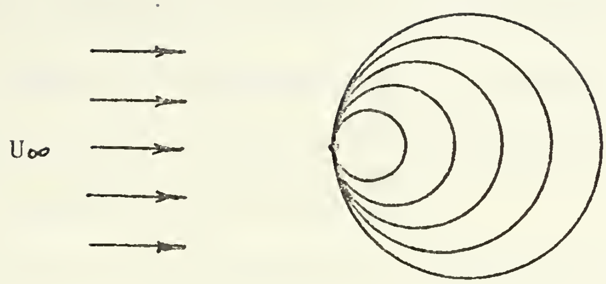
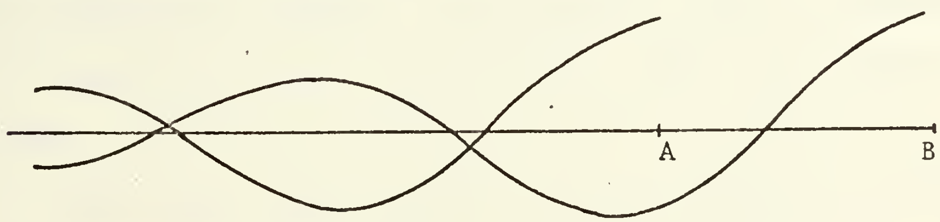


Figure 3. TRANSONIC WAVE PROPAGATION

Assuming now a continuous spatial distribution of disturbance sources, it is readily understood from Fig. 4 that the receding waves will tend to cancel provided the wavelength (given by $2\pi|a-U|/\omega$, where ω is the circular frequency of the disturbance) is very short compared to body reference length, C. However, if the sources oscillate very slowly,



Source A and B with Respective Waves

The Resulting Wave

Figure 4. WAVE CANCELLATION

pressure waves of the same sign will have sufficient time to interact and possibly to build up to finite amplitudes. Obviously, these receding waves may create large phase lags between motion and pressure on the airfoil causing severe aeroelastic instability problems.

We can draw the conclusion, therefore, that sufficiently unsteady transonic flows can be analyzed by linearized theory, Eqn. (II-14), whereas only mildly unsteady transonic flows must be investigated using Eqn. (II-25). The condition for linearization is generally stated as [Ref. 12]

$$K \gg |1 - M_{LOCAL}|$$

where $|1 - M_{LOCAL}|$ is the largest deviation of the Mach number from unity in the flow and K is the reduced frequency ($K = \omega C/U_\infty$).

B. DEVELOPMENT OF THE BOUNDARY CONDITIONS

To completely describe the problem, proper boundary and initial conditions must be specified. The most important ones are:

1. The flow on the surface of the body must be tangent at any instant of time.

2. Sommerfeld's radiation condition must be satisfied at any instant of time, i.e., waves propagate away from sources of disturbance.

Further, certain edge and wake conditions must be satisfied, e.g., smooth flow at trailing edges must be established (Kutta condition).

The general flow tangency condition can be stated as follows. If the equation of the surface of a body which is moving in a time dependent manner is given by

$$F(X,Y,t) = 0, \tag{II-26}$$

then the flow tangency condition requires that the total derivative of the function F is zero, i.e.,

$$\frac{DF}{Dt} = F_t + U\, F_x + V\, F_y = 0\ . \tag{II-27}$$

Consider an airfoil that is placed in a Cartesian coordinate system in such a way that if the airfoil is restricted to small amplitude motions out of the Y = 0 plane, then its upper and lower surface can be described by

$$\begin{aligned} F_u &= Y - Y_u(X,t) = 0 \\ F_L &= Y - Y_L(X,t) = 0 \end{aligned} \tag{II-28}$$

Therefore, from the boundary condition one has

$$\begin{aligned} V &= Y_{u_t} + U\, Y_{u_x} \qquad \text{at} \quad Y = Y_u(X,t) \\ V &= Y_{L_t} + U\, Y_{L_x} \qquad \text{at} \quad Y = Y_L(X,t) \end{aligned} \tag{II-29}$$

One now introduces the small perturbation assumption,

$$\Phi = U_\infty X + \phi \tag{II-15}$$

where the disturbance velocity components are

$$\begin{aligned} u &= \phi_x \\ v &= \phi_y \end{aligned} \tag{II-30}$$

such that

$$U = U_\infty + u \tag{II-31}$$

$$V = v \tag{II-32}$$

and $u, v \ll U_\infty$

In (II-29) the term, $u\, Y_{u_x}$, can then be neglected compared to the much larger term, $U_\infty Y_{u_x}$. The boundary conditions then simplify to

$$v = Y_{u_t} + U_\infty Y_{u_x} \quad \text{for } Y = Y_u(X,t)$$

and

$$v = Y_{L_t} + U_\infty Y_{L_x} \quad \text{for } Y = Y_L(X,t). \tag{II-33}$$

For small amplitude oscillations, (II-33) can be further simplified by expressing the flow tangency condition in the plane Y = 0. Taylor series expansion gives

$$v(X,Y_u,t) = v(X,0^+,t) + Y_u \frac{\partial v(X,0^+,t)}{\partial Y} + \ldots.$$

and

$$v(X,Y_L,t) = v(X,0^-,t) + Y_L \frac{\partial v(X,0^-,t)}{\partial Y} + \ldots.$$

Using the same argument as above, the product terms can be neglected so that the flow tangency condition can be stated as

$$v = Y_{u_t} + U_\infty\, Y_{u_x} \qquad \text{for } Y = 0^+$$

$$v = Y_{L_t} + U_\infty\, Y_{L_x} \qquad \text{for } Y = 0^- \tag{II-34}$$

For a thin foil performing a time dependent motion about the Y = 0 plane, the airfoil thickness terms can be disregarded by similar arguments. The above conditions become

$$v = \frac{\partial h}{\partial t} + U_\infty \frac{\partial h}{\partial x} \qquad \text{for} \quad Y = 0 \qquad \text{(II-35)}$$

where h(X,t) describes the time dependent chord line of the blade.

C. THE DERIVATION OF THE UNSTEADY EQUATIONS

Define the following nondimensional terms.

$$\begin{aligned} \overline{X} &= \frac{X}{C} \\ \overline{Y} &= \frac{Y}{C} \\ \overline{t} &= \frac{t\ U_\infty}{C} \\ \overline{\phi} &= \frac{\phi}{U_\infty C} \\ \overline{v} &= \frac{v}{U_\infty} \\ \overline{h} &= \frac{h}{C} \end{aligned} \qquad \text{(II-36)}$$

where C is the chord of the foil. Equations (II-25) and (II-35) may now be written in nondimensional form:

$$[1 - M_\infty^2 - (\gamma+1)M_\infty^2\overline{\phi}_{\overline{x}}]\overline{\phi}_{\overline{xx}} + \overline{\phi}_{\overline{yy}} - 2M_\infty^2\overline{\phi}_{\overline{xt}} - M_\infty^2\overline{\phi}_{\overline{tt}} = 0 \qquad \text{(II-37)}$$

and $$\overline{v} = \overline{h}_{\overline{t}} + \overline{h}_{\overline{x}} \qquad \text{(II-38)}$$

Since the main interest of this thesis centers on the determination of the aeroelastic stability of a foil, only small perturbations from the steady state need be studied.

If one restricts oneself to such small amplitude perturbations, it is possible to write the perturbation potential as the sum of a steady state and a time dependent potential. Also, in the case of a harmonic oscillation, one may describe the time dependent quantities as the product of a complex amplitude function and a harmonic time factor. Therefore one defines the potentials ϕ, and ψ such that

$$\overline{\phi}(\overline{X},\overline{Y},\overline{t}) = \phi_1(\overline{X},\overline{Y}) + \psi(\overline{X},\overline{Y},K)e^{iK\overline{t}} \tag{II-39}$$

and the foil chord function and vertical velocity are

$$\overline{h}(\overline{X},\overline{t}) = h_1(\overline{X}) + \hat{h}(\overline{X},K)e^{iK\overline{t}} \tag{II-40}$$

$$\overline{v}(\overline{X},\overline{t}) = v_1(\overline{X}) + \hat{v}(\overline{X},K)e^{iK\overline{t}} \tag{II-41}$$

where
$$K = \frac{\omega C}{U_\infty} \tag{II-42}$$

It may be shown that (II-37) becomes

$$[1-M_\infty^2-M_\infty^2(\gamma+1)\phi_{1\overline{x}}]\phi_{1\overline{xx}} + \phi_{1\overline{yy}} = 0 \tag{II-43}$$

which is the steady state equation, and the unsteady equation is

$$[1-M_\infty^2]\psi_{\overline{xx}} - M_\infty^2(\gamma+1)\frac{\partial}{\partial\overline{x}}(\phi_{1\overline{x}}\psi_{\overline{x}}) + M_\infty^2K^2\psi$$

$$-2iKM_\infty^2\psi_{\overline{x}} + \psi_{\overline{yy}} = 0 . \tag{II-44}$$

The boundary equation (II-38) becomes

$$v_1(\overline{X}) = h_{1\overline{x}}(\overline{X}) \tag{II-45}$$

and
$$\hat{v}(\overline{X},K) = \hat{h}_{\overline{x}}(\overline{X},K) + iK\hat{h}(\overline{X},K) \tag{II-46}$$

On the foil surface equation (II-43) must satisfy the boundary equation (II-45); (II-44) must satisfy (II-46).

For convenience, the overbar and hat will now be dropped and the steady and unsteady equations with boundary equations become

$$[1-M_\infty^2 - M_\infty^2(\gamma+1)\phi_{1x}]\phi_{1xx} + \phi_{1yy} = 0 \qquad \text{(II-47)}$$

subject to

$$v_1 = h_{1x} \quad , \qquad \text{at } Y=0 \qquad \text{(II-48)}$$

and

$$[1-M_\infty^2]\psi_{xx} - M_\infty^2(\gamma+1)\frac{\partial}{\partial x}(\phi_{1x}\psi_x) + M_\infty^2 K^2 \psi$$

$$- 2iKM_\infty^2\psi_x + \psi_{yy} = 0 \qquad \text{(II-49)}$$

subject to

$$v = h_x + iKh \quad , \quad \text{at } Y=0 \ . \qquad \text{(II-50)}$$

Note that (II-47) is a nonlinear equation in ϕ_1. Equation (II-49) is linear in ψ but the existence of the variable coefficient ϕ_{1x} considerably complicates its solution.

D. A BRIEF REVIEW OF POSSIBLE APPROACHES TO UNSTEADY TRANSONIC FLOW ANALYSIS

The following is a presentation of various approaches to the solution of equation (II-49).

1. Method of Teipel and Hosokawa

An approximation of equation (II-49) was proposed by Teipel [Ref. 9] and Hosokawa [Ref. 13] for oscillating

airfoils in sonic flow ($M_\infty = 1$). For steady sonic flow, Oswatitch [Ref. 8] introduced a linearized approximation by setting

$$(\gamma+1)\phi_{1xx} = \lambda = \text{constant}, \quad \lambda > 0 \ , \qquad \text{(II-51)}$$

allowing (II-43) to be written in the easily solved form

$$\phi_{1yy} - \lambda\, \phi_{1x} = 0 \ . \qquad \text{(II-52)}$$

Teipel and Hosokawa generalized this procedure to unsteady sonic flow by approximating equation (II-49) with (neglecting the term in ψ_{xx})

$$-\lambda\, \psi_x + \psi_{yy} - 2iK\psi_x + K^2\psi = 0 \ . \qquad \text{(II-53)}$$

Defining Q

$$Q = \lambda + 2iK \ , \qquad \text{(II-54)}$$

one has from (II-46)

$$\psi_{yy} - Q\, \psi_x + K^2\, \psi = 0 \ . \qquad \text{(II-55)}$$

This equation allows the study of sonic flow past a single oscillating airfoil in unbounded flow over the entire frequency range. It contains the steady solution (Oswatitch approximation) as a special case. For large K (Refer to page 23) the term containing ϕ_{1xx} becomes negligibly small; therefore, retention of λ at high reduced frequencies becomes unnecessary. Thus, when one takes $\lambda = 0$, the Teipel-Hosokawa theory also contains the linearized theory.

2. Local Linearization Technique for Steady Transonic Flow

It was shown that the Oswatitch approximation of the steady transonic small perturbation equation is

$$\phi_{1yy} - \lambda \phi_{1x} = 0 , \qquad \lambda = \text{constant} > 0 . \tag{II-52}$$

The solution to this equation for sonic flow past a single airfoil is

$$\phi_1(X,Y) = - \frac{1}{\sqrt{\pi\lambda}} \int_{-1}^{X} \frac{V(S)}{\sqrt{X-S}} e^{-\frac{\lambda Y^2}{4(X-S)}} ds . \tag{II-56}$$

Of particular interest is the velocity <u>on</u> the profile which is given by

$$\phi_{1x}(X,0) = - \frac{1}{\sqrt{\pi\lambda}} \frac{d}{dx} \int_{-1}^{X} \frac{V(S)}{\sqrt{X-S}} ds . \tag{II-57}$$

Recalling the definition of λ, (II-51), Spreiter argued that the original variable, ϕ_{1xx}, could be reintroduced into equation (II-57) thus linearizing the equation locally. This leads to an ordinary nonlinear differential equation in ϕ_{1x} which can be solved analytically. The agreement with experiment is remarkably good [Ref. 10]. Its application to the choked wind tunnel wall interference problem was given by Sandeman [Ref. 4]. It is expected that an extension of this technique to the unsteady case may prove fruitful.

3. WKBJ - Method

Rubbert [Ref. 14] applied the WKBJ method to this problem. Differentiating the axisymmetric counterpart of

equation (II-49) for $M_\infty = 1$, multiplying by ϕ_{1x} and applying the Hankel transform, the equation can be reduced to the form

$$\Psi_{XX} + N^2 q(X,N)\Psi = 0 \qquad \text{(II-58)}$$

where N is a large parameter. Unfortunately no inversion of the resulting transform integrals has been found.

4. Method of Geometrical Acoustics

A third approach has recently been proposed to obtain the thickness effects for oscillating wings in transonic flow [Ref. 15]. This procedure begins with a given non-uniform steady transonic flow field which has been determined either analytically or experimentally (In the latter case an analytical expression must be obtained which matches the measured pressure coefficient.). Superimposing upon this flow field small unsteady perturbations, the path and transmission time of these acoustic perturbations are determined by the methods of geometrical acoustics. Again, only preliminary calculations have been carried out to date.

5. Unsteady Transonic Flow with Shock Waves

Typically, transonic flows around an airfoil exhibit locally supersonic regions which are terminated by shock waves. Often, the interactions of small unsteady perturbations with these shock waves may be responsible for aerodynamically induced instabilities. Eckhaus [Ref. 16] made the problem mathematically tractable by assuming a normal shock extending from the surface to infinity with uniform

steady supersonic flow in front and uniform steady subsonic flow behind the shock. Performing a small perturbation analysis, i.e., superimposing small time dependent perturbations upon this idealized steady transonic flow, he was able to obtain equations for the interaction of the sound waves with the shock. However, only a few numerical results were obtained.

For the following study, the Teipel-Hosokawa approach was used.

III. TRANSONIC UNSTEADY CASCADE FLOW PROBLEM

A. PROBLEM FORMULATION FOR SONIC FLOW PAST OSCILLATING UNSTAGGERED CASCADES

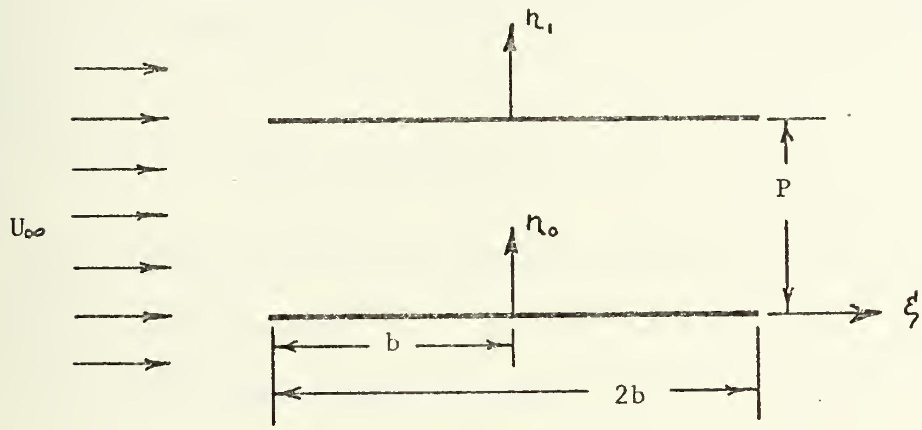


Figure 5. UNSTAGGERED CASCADE

Consider an infinite cascade of thin blades arranged as shown in Figure 5. Blade chord is 2b and blade spacing is P. Subscript zero referes to the reference blade and subscript one refers to the adjacent blade in the positive ordinate (η) direction. The reference coordinate system is fixed at mid-chord with the abscissa (ξ) in the direction of flow.

Two η coordinates are specified with η_o originating on the reference blade and η_1 on the adjacent blade such that

$$\eta_1 = \eta_o + P \quad . \tag{III-1}$$

Now the following nondimensional variables are defined:

$$X = \frac{\xi}{b} \tag{III-2}$$

$$Y_m = \frac{\eta_m}{b}, \quad m=0,1 \tag{III-3}$$

$$R = \frac{P}{b} \tag{III-4}$$

As the governing equation, the Teipel approximation is used,

$$\psi_{yy} - Q\,\psi_x + K^2\,\psi = 0\ . \tag{II-55}$$

This equation must be solved subject to the following boundary equations:

1. Zero disturbance upstream of the cascade.
2. Flow tangency on the blades.

As shown in Section II-B, this latter requirement can be expressed mathematically as follows. Along the reference blade equation (II-50) may be written

$$\psi_y(X, Y_o=0) = v_o = \frac{\partial h_o}{\partial X} + iKh_o \tag{III-5}$$

and along the adjacent blade

$$\psi_y(X, Y_o=R) = v_1 = \frac{\partial h_1}{\partial X} + iKh_1\ . \tag{III-6}$$

To solve this boundary value problem, the potential function ψ is written as the summation

$$\psi(X, Y_o) = \Phi_o^o(X, Y_o) + \Phi_o(X, Y_o) + \Phi_1^o(X, Y_1)$$

$$+\ \Phi_1(X, Y_1) \tag{III-7}$$

where Φ_o^o and Φ_1^o are the isolated blade solutions for the reference and adjacent blades and Φ_o and Φ_1 are interference potentials.

Such an approach has previously been proposed for the subsonic and supersonic cases by D. N. Gorelov [Refs. 3 and 17]. Also refer to Samoylovich [Ref. 18].

Inserting (III-7) into (III-5) and (III-6) and realizing

$$\Phi^o_{my}(X,0) = v_m \ , \quad m=0,1 \ ,$$

one obtains for the reference blade,

$$\Phi_{oy}(X,Y_o= 0) + \Phi^o_{1y}(X,Y_1= -R) + \Phi_{1y}(X,Y_1= -R) = 0 \tag{III-8}$$

and for the adjacent blade,

$$\Phi^o_{oy}(X,Y_o= R) + \Phi_{oy}(X,Y_o= R) + \Phi_{1y}(X,Y_1= 0) = 0 \ . \tag{III-9}$$

The isolated blade solution is already known from Teipel [Ref. 9] as

$$\Phi^o_o(X,Y_o) = -\frac{1}{\sqrt{\pi Q}} \int_{-1}^{X} v_o(S) \frac{\mathrm{EXP}[\frac{K^2}{Q}(X-S) - \frac{Q}{4}\frac{Y_o^2}{X-S}]}{\sqrt{X-S}} \, dS \tag{III-10}$$

and

$$\Phi^o_1(X,Y_1) = \frac{1}{\sqrt{\pi Q}} \int_{-1}^{X} v_1(S) \frac{\mathrm{EXP}[\frac{K^2}{Q}(X-S) - \frac{Q}{4}\frac{Y_1^2}{X-S}]}{\sqrt{X-S}} \, dS \tag{III-11}$$

The interference potentials must have the same form but contain unknown source distributions u_o and u_1; thus,

$$\Phi_o(X,Y_o) = -\frac{1}{\sqrt{\pi Q}} \int_{-1}^{X} u_o(S) \frac{\mathrm{EXP}[\frac{K^2}{Q}(X-S) - \frac{Q}{4}\frac{Y_o^2}{X-S}]}{\sqrt{X-S}} \, dS \tag{III-12}$$

and

$$\Phi_1(X,Y_1) = \frac{1}{\sqrt{\pi Q}} \int_{-1}^{X} u_1(S) \frac{\mathrm{EXP}[\frac{K^2}{Q}(X-S) - \frac{Q}{4}\frac{Y_1^2}{X-S}]}{\sqrt{X-S}} dS. \qquad \text{(III-13)}$$

These source distributions are approximated by the power series as follows:

$$u_m = \sum_{N=0}^{L-1} \vartheta_{mN} X^N \qquad \text{(III-14)}$$

and

$$v_m = \sum_{N=0}^{L2} \theta_{mN} X^N . \qquad \text{(III-15)}$$

Inserting these approximations, (III-14) and (III-15), into equations (III-8) and (III-9) results in

$$\sum_{N=0}^{L-1}\left\{ \vartheta_{oN}X^N + \frac{R}{2}\sqrt{\frac{Q}{\pi}}\,(\theta_{1N}+\vartheta_{1N}) \int_{-1}^{X} \frac{S^N}{(X-S)^{3/2}}\, \mathrm{EXP}[\frac{K^2}{Q}(X-S)-\frac{Q}{4}\frac{R^2}{X-S}]dS\right\}$$
$$= 0 \qquad \text{(III-16)}$$

for the reference blade and

$$\sum_{N=0}^{L-1}\left\{\vartheta_{1N}X^N + \frac{R}{2}\sqrt{\frac{Q}{\pi}}(\theta_{oN}+\vartheta_{oN}) \int_{-1}^{X} \frac{S^N}{(X-S)^{3/2}}\, \mathrm{EXP}[\frac{K^2}{Q}(X-S)-\frac{Q}{4}\frac{R^2}{X-S}]dS\right\}$$
$$= 0 \qquad \text{(III-17)}$$

for the adjacent blade.

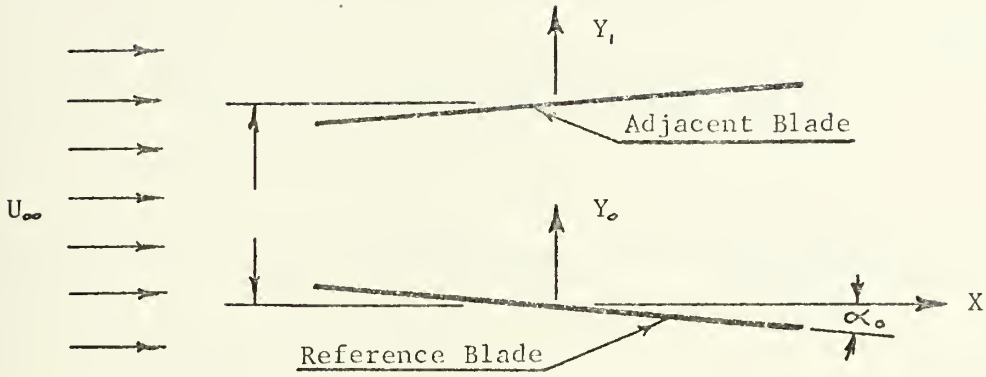


Figure 6. BOUNDARY CONDITIONS FOR π PHASE LAG

The coefficients in (III-15) may be evaluated for the case of simple oscillation, no translation, and π phase lag between blades as follows. One arbitrarily sets the angle of maximum amplitude for the reference blade at $\alpha_o = -1$ [Refer to Fig. 6]. Then the angle of the adjacent blade is $\alpha_1 = 1$. For small α, equation (III-5) may be written

$$\psi_y(X, Y_o = 0) = -1 - iKX \qquad \text{(III-18)}$$

and (III-6),

$$\psi_y(X, Y_1 = 0) = 1 + iKX \, . \qquad \text{(III-19)}$$

Hence, the coefficients are

$$\begin{aligned} \theta_{oo} &= -1 \\ \theta_{o1} &= -iK \\ \theta_{1o} &= 1 \end{aligned} \tag{III-20}$$

and

$$\theta_{11} = iK \; .$$

Similarly, for in phase oscillation, the coefficients are

$$\theta_{oo} = \theta_{1o} = -1 \tag{III-21}$$

and

$$\theta_{o1} = \theta_{11} = -iK \; .$$

Satisfying boundary equations (III-16) and (III-17) at L points along the blades generates a set of 2L equations in 2L unknown interference coefficients. The coefficients in hand, $\psi(X,Y_o)$ in (III-7) becomes a known function.

A computer program for the solution of this problem in the π phase shift case is developed in Appendix A. Some sample results are presented in Table II for the steady case and Table VII for the unsteady case.

B. EVALUATION OF THE INTEGRALS OCCURRING IN EQUATIONS (III-16) and (III-17)

The integrals occurring in (III-16) and (III-17) are of the form

$$I_N = \int_{-1}^{X} \frac{S^N}{(X-S)^{3/2}} \, \mathrm{EXP}\left[\frac{K^2}{Q}(X-S) - \frac{Q}{4}\,\frac{R^2}{(X-S)}\right] dS \quad . \tag{III-22}$$

By making the substitution, $U = X-S$, one obtains

$$I_N = \int_{0}^{X+1} \frac{(X-U)^N}{U^{3/2}} \, \mathrm{EXP}\left[\frac{K^2}{Q}\,U - \frac{Q}{4}\,\frac{R^2}{U}\right] dU \tag{III-23}$$

Recalling $Q = \lambda + 2iK$, I_N is complex and must be resolved into real and imaginary parts for evaluation.

The exponential term in (III-23) may be written

$$\mathrm{EXP}\left[\frac{K^2}{Q} U - \frac{Q}{4} \frac{R^2}{U}\right] = \mathrm{EXP}\left[\frac{\lambda K^2 U}{\lambda^2 + 4K^2} - \frac{\lambda R^2}{4U}\right]$$

$$\cdot \mathrm{EXP}\left\{ i\left[-\frac{2K^3 U}{\lambda^2 + 4K^2} - \frac{KR^2}{2U}\right]\right\} \quad . \qquad \text{(III-24)}$$

One now defines the following terms:

$$A = \frac{K^2}{\lambda^2 + 4K^2}$$

$$B = \frac{\lambda R^2}{4}$$

$$E = \frac{KR^2}{2}$$

Equation (III-24) may be rewritten,

$$\mathrm{EXP}\left[\frac{K^2}{Q} U - \frac{Q}{4} \frac{R^2}{U}\right] = \mathrm{EXP}\left[\lambda AU - \frac{B}{U}\right]\mathrm{EXP}\left\{i\left[-2KAU - \frac{E}{U}\right]\right\} \qquad \text{(III-25)}$$

Separating (III-25) into its real and imaginary parts, one obtains the following two fundamental integrals from (III-23):

$$I_{NR} = \int_0^{x+1} \frac{(X-U)^N}{U^{3/2}} \mathrm{EXP}\left[\lambda AU - \frac{B}{U}\right] \mathrm{COS}\left[2KAU \pm \frac{E}{U}\right] dU \qquad \text{(III-26)}$$

and $$I_{NI} = \int_0^{x+1} \frac{(X-U)^N}{U^{3/2}} \mathrm{EXP}\left[\lambda AU - \frac{B}{U}\right] \mathrm{SIN}\left[2KAU + \frac{E}{U}\right] dU \qquad \text{(III-27)}$$

where $I_N = I_{NR} - iI_{NI}$ (III-28)

Expanding $(X-U)^N$ in the finite binomial series,

$$(X-U)^N = \sum_{J=1}^{N+1} \frac{(-1)^{J-1} \; N! \; X^{N-J+1} \; U^{J-1}}{(J-1)! \; (N-J+1)!} \tag{III-29}$$

equations (III-26) and (III-27) become

$$I_{NR} = \sum_{J=1}^{N+1} \frac{(-1)^{J-1} \; N! \; X^{N-J+1}}{(J-1)! \; (N-J+1)!} \int_0^{X+1} \left\{ U^{J-\frac{5}{2}} \, EXP[\lambda AU - \frac{B}{U}] \cdot \right.$$

$$\left. \cdot \, COS[2KAU + \frac{E}{U}] \right\} dU \tag{III-30}$$

and

$$I_{NI} = \sum_{J=1}^{N+1} \frac{(-1)^{J-1} \; N! \; X^{N-J+1}}{(J-1)! \; (N-J+1)!} \int_0^{X+1} \left\{ U^{J-\frac{5}{2}} \, EXP[\lambda AU - \frac{B}{U}] \cdot \right.$$

$$\left. \cdot \, SIN[2KAU + \frac{E}{U}] \right\} dU \tag{III-31}$$

The fundamental integrals in (III-30) and (III-31) do not lend themselves to analytical integration. Numerical integration techniques may be used but care must be exercised in the vicinity of the lower limit. Here the integrand is singular when $J < 5/2$. The problem is further aggravated by the rapidly oscillatory character of the functions $SIN[\lambda AU - \frac{B}{U}]$ and $COS[\lambda AU - \frac{B}{U}]$.

Three specific cases are posed for the treatment of these integrals.

1. The Steady Case

Setting $K = 0$ reduces (III-30) and (III-31) to the following:

$$I_{NR} = \sum_{J=1}^{N+1} \frac{(-1)^{J-1}\ N!\ X^{N-J+1}}{(J-1)!\ (N-J+1)!} \int_0^{X+1} U^{J-\frac{5}{2}} \mathrm{EXP}[-\frac{B}{U}]\, dU \qquad \text{(III-32)}$$

and

$$I_{NI} = 0$$

The numerical integration of this case is straightforward. The exponential term forces the integrand to zero near the origin permitting one to neglect that portion of the integral very near U = 0. Equation (III-32) may also be evaluated by expressing it in terms of the error function. (Note: complete agreement between the two integration techniques has been obtained thus verifying the numerical integration procedure for the steady case.)

2. <u>The Oscillatory Case with $\lambda = 0$</u>

In the unsteady case for $\lambda = 0$, equations (III-30) and (III-31) contain the basic integrals

$$\int_0^{X+1} U^{J-\frac{5}{2}} \mathrm{COS}[2KAU + \frac{E}{U}]\, dU \qquad \text{(III-33)}$$

and

$$\int_0^{X+1} U^{J-\frac{5}{2}} \mathrm{SIN}[2KAU + \frac{E}{U}]\, dU. \qquad \text{(III-34)}$$

To avoid the previously mentioned difficulties near the lower limit, the integration must be accomplished in two steps, i.e.,

$$\int_0^{X+1} U^{J-\frac{5}{2}} \frac{\text{COS}}{\text{SIN}}[2KAU + \frac{E}{U}]dU = \int_0^{a} U^{J-\frac{5}{2}} \frac{\text{COS}}{\text{SIN}}[2KAU + \frac{E}{U}]dU +$$

$$+ \int_a^{X+1} U^{J-\frac{5}{2}} \frac{\text{COS}}{\text{SIN}}[2KAU + \frac{E}{U}]dU. \quad \text{(III-35)}$$

For judicious selection of (a), the second integral in the RHS may be numerically integrated in a straightforward manner. The integral on the limits 0 to (a), which will henceforth be referred to as the correction integral, may be approximated for sufficiently small (a) as follows.

Recall the following relations:

$$\text{COS}[2KAU + \frac{E}{U}] = \text{COS}[2KAU]\text{COS}[\frac{E}{U}] - \text{SIN}[2KAU]\text{SIN}[\frac{E}{U}] \quad \text{(III-36)}$$

and

$$\text{SIN}[2KAU + \frac{E}{U}] = \text{SIN}[2KAU]\text{COS}[\frac{E}{U}] + \text{COS}[2KAU]\text{SIN}[\frac{E}{U}] \quad \text{(III-37)}$$

and the truncated series expansions

$$\text{COS}[2KAU] = 1 - \frac{(2KAU)^2}{2} \quad \text{(III-38)}$$

and

$$\text{SIN}[2KAU] = 2KAU - \frac{(2KAU)^3}{6}\,. \quad \text{(III-39)}$$

Equations (III-38) and (III-39) are valid for 2KAU sufficiently close to zero. One may now write

$$\int_0^a U^{J-\frac{5}{2}} \cos[2KAU + \frac{E}{U}]dU =$$

$$\int_0^a U^{J-\frac{5}{2}} \cos[\frac{E}{U}]dU - (2KA)\int_0^a U^{J-\frac{3}{2}} \sin[\frac{E}{U}]dU$$

$$- \frac{(2KA)^2}{2} \int_0^a U^{J-\frac{1}{2}} \cos[\frac{E}{U}]dU$$

$$+ \frac{(2KA)^3}{6} \int_0^a U^{J+\frac{1}{2}} \sin[\frac{E}{U}]dU \qquad \text{(III-40)}$$

and

$$\int_0^a U^{J-\frac{5}{2}} \sin[2KAU + \frac{E}{U}]dU =$$

$$\int_0^a U^{J-\frac{5}{2}} \sin[\frac{E}{U}]dU + (2KA) \int_0^a U^{J-\frac{3}{2}} \cos[\frac{E}{U}]dU$$

$$- \frac{(2KA)^2}{2} \int_0^a U^{J-\frac{1}{2}} \sin[\frac{E}{U}]dU$$

$$- \frac{(2KA)^3}{6} \int_0^a U^{J+\frac{1}{2}} \cos[\frac{E}{U}]dU \ . \qquad \text{(III-41)}$$

In theory integrals of the form

$$\int_0^a U^{m/2} \begin{matrix}\cos \\ \sin\end{matrix} [\frac{E}{U}]dU$$

may be expressed in terms of the Fresnel integrals. Hence, integration of (III-40) and (III-41) can be accomplished in "closed" form. (In actuality there is a computer accuracy limit on this technique as described in Appendix A-IV.)

Verification of this technique for J = 1 was accomplished by checking it against a known analytical function as follows:

$$\int_0^a \frac{1}{U^{3/2}} \begin{matrix}\mathrm{COS}\\\mathrm{SIN}\end{matrix}[2KAU + \frac{E}{U}]dU + \int_a^\infty \frac{1}{U^{3/2}} \begin{matrix}\mathrm{COS}\\\mathrm{SIN}\end{matrix}[2KAU + \frac{E}{U}]dU =$$

$$\int_0^\infty \frac{1}{U^{3/2}} \begin{matrix}\mathrm{COS}\\\mathrm{SIN}\end{matrix}[2KAU + \frac{E}{U}]dU \qquad \text{(III-42)}$$

where the correction integral was evaluated using the Fresnel integral technique, the second integral was evaluated using numerical integration techniques, and the integral on the RHS was evaluated using the following relation:

$$\int_0^\infty \frac{1}{U^{3/2}} \begin{matrix}\mathrm{COS}\\\mathrm{SIN}\end{matrix}[gU + \frac{b}{U}]dU = \sqrt{\pi/b}\, \begin{matrix}\mathrm{COS}\\\mathrm{SIN}\end{matrix}[\frac{\pi}{4} + \sqrt{2gb}\,] \,. \qquad \text{(III-43)}$$

For the case K = 0.01 and a = 0.01, Table IA presents the comparison of the value of (III-43) with that obtained from (III-42) for various R's. (The numerical integration was truncated at an upper limit of 500 and the convergence criterion EPS for subroutine ZSUM was set at 1.0E-4.)

Table IB displays the computed integral using (III-35) with an upper limit of X+1 = 2.0 for various values of (a). (The convergence limit placed on ZSUM was set at 1.0E-5.)

Table variables are listed below.

DK = K

DL = λ

RHO = R

AL = a (Correction integral upper limit.)

SCOR = Value of correction integral (III-41) for J = 1.

CCOR = Value of correction integral (III-40) for J = 1.

SUMI = Value of numerical integration term from (a) to specified upper limit for the SIN integral.

SUMR = Value of numerical integration term from (a) to specified upper limit for the COS integral.

SINT = Value of (III-35) SIN integral for specified upper limit and J = 1.

CINT = Value of (III-35) COS integral for specified upper limit and J = 1.

SSIN = Value of (III-43) SIN integral.

CCOS = Value of (III-43) COS integral.

3. The General Case, $K \neq 0$ and $\lambda \neq 0$

In the apparent absence of a suitable correction integral for this case, one must evaluate the full equations (III-26) and (III-27) by numerical methods. The difficulty close to the lower limit is alleviated to some extent by the exponential term. The exponential function drives the amplitude of the integrand to zero more rapidly than a negative power of U can amplify the highly oscillatory behavior of the transcendental functions. For this reason, as in the case of K = 0, one may justify neglecting that portion of the integrals very near zero.

Although no results are presented, the computer routine developed in Appendix A is designed to carry out such a case.

TABLE IA VERIFICATION OF CORRECTION INTEGRAL

R	SCOR	SUMI	SINT	SSIN
0.5	34.62	1.016	35.63	35.57
1.0	14.45	3.458	17.91	17.85
1.5	4.968	7.032	12.00	11.94
2.0	-1.114	10.16	9.045	8.987

R	CCOR	SUMR	CINT	CCOS
0.5	15.48	19.80	35.28	35.32
1.0	-1.782	19.34	17.56	17.60
1.5	-5.797	17.45	11.65	11.69
2.0	-4.490	13.19	8.696	8.736

COMPARISON OF VALUES COMPUTED USING
EQUATION (III-43) WITH THOSE OBTAINED
USING (III-42) FOR AN UPPER LIMIT OF 500.

TABLE IB VERIFICATION OF CORRECTION INTEGRAL

```
DK= 0.500       DL= 0.0         RHO= 0.500      AL= 0.010
SCOR=  1.57005D 00     CCOR=  1.67114D-01
SUMI=  3.96851D 00     SUMR=  3.24013D 00
SINT=  5.53856D 00     CINT=  3.40724D 00

DK= 0.500       DL= 0.0         RHO= 0.500      AL= 0.050
SCOR=  1.66484D 00     CCOR= -2.65483D 00
SUMI=  3.87372D 00     SUMR=  6.06207D 00
SINT=  5.53856D 00     CINT=  3.40724D 00

DK= 0.500       DL= 0.0         RHO= 0.500      AL= 0.100
SCOR=  3.74310D 00     CCOR= -1.12801D 00
SUMI=  1.79545D 00     SUMR=  4.53526D 00
SINT=  5.53856D 00     CINT=  3.40724D 00

DK= 0.500       DL= 0.0         RHO= 0.500      AL= 0.200
SCOR=  4.62026D 00     CCOR=  4.96270D-01
SUMI=  9.18297D-01     SUMR=  2.91097D 00
SINT=  5.53856D 00     CINT=  3.40724D 00

DK= 0.500       DL= 0.0         RHO= 0.500      AL= 0.300
SCOR=  4.87826D 00     CCOR=  1.27508D 00
SUMI=  6.60294D-01     SUMR=  2.13216D 00
SINT=  5.53856D 00     CINT=  3.40724D 00
```

C. CALCULATION OF THE POTENTIAL FUNCTION AND THE PRESSURE COEFFICIENT

The potential function is obtained from equation (III-7),

$$\psi(X,Y_o) = \Phi_o^o(X,Y_o) + \Phi_o(X,Y_o) + \Phi_1^o(X,Y_1) + \Phi_1(X,Y_1) \tag{III-7}$$

The interference potentials Φ_o and Φ_1 may now be computed using the previously determined coefficients ($\mathcal{D}_{mN}$, $m = 0,1$; $N = 0,\ldots,L-1$) in equations (III-12) and (III-13). Equation (III-7) may be rewritten in terms of the coefficients as follows:

$$\psi(X,Y_o) = \sum_{N=0}^{L-1} \left\{ -\frac{1}{\sqrt{\pi Q}} (\theta_{oN}+\mathcal{D}_{oN}) \int_{-1}^{X} \frac{S^N}{\sqrt{X-S}} \, \mathrm{EXP}\left[\frac{K^2}{Q}(X-S) - \frac{Q}{4} \frac{Y_o^2}{(X-S)}\right] dS \right.$$

$$\left. + \frac{1}{\sqrt{\pi Q}} (\theta_{1N}+\mathcal{D}_{1N}) \int_{-1}^{X} \frac{S^N}{\sqrt{X-S}} \, \mathrm{EXP}\left[\frac{K^2}{Q}(X-S) - \frac{Q}{4} \frac{Y_1^2}{(X-S)}\right] dS \right\} \tag{III-44}$$

The resulting integrals may be evaluated using the same techniques utilized in the coefficient derivation.

The pressure coefficient is obtained from the relation

$$Cp = -2(\psi_X + iK\psi) \; . \tag{III-45}$$

By placing the partial with respect to X of equation (III-44) into (III-45), one may obtain the pressure distribution at any point in the flow field as well as on the blade surfaces. Further, the force and moment coefficients are easily obtained by integration over the blade chord.

D. EVALUATION OF THE STEADY STATE POTENTIAL ALONG THE BLADE

The computation of the steady state potential along the upper surface of the reference blade follows from equation (III-44) evaluated at $K = 0$, $Y_o = 0$, and $Y_1 = -R$. The development which follows is for the π phase shift case where [From Eqns. (III-20).]

$$\begin{aligned} \theta_{oo} &= -1 \\ \theta_{1o} &= 1 \end{aligned} \tag{III-46}$$

Equation (III-44) becomes

$$\begin{aligned} \psi_{K=0}(X,0) = {} & \frac{1}{\sqrt{\pi\lambda}}\left\{ \int_{-1}^{X} \frac{1}{\sqrt{X-S}}\, dS + \int_{-1}^{X} \frac{\mathrm{EXP}[-\frac{\lambda}{4}\frac{R^2}{X-S}]}{\sqrt{X-S}}\, dS \right\} \\ & + \frac{1}{\sqrt{\pi\lambda}} \sum_{N=0}^{L-1} \left\{ \mathcal{V}_{1N} \int_{-1}^{X} \frac{S^N}{\sqrt{X-S}} \mathrm{EXP}[-\frac{\lambda}{4}\frac{R^2}{X-S}] dS \right. \\ & \left. - \mathcal{V}_{oN} \int_{-1}^{X} \frac{S^N}{\sqrt{X-S}}\, dS \right\} . \end{aligned} \tag{III-47}$$

The computer program is documented in Appendix B-I. The resulting potential values are compared with those obtained from the Sandeman wall interference solution of Section III-E in Tables III and IV.

E. LAPLACE TRANSFORM SOLUTION FOR THE SONIC WIND TUNNEL WALL INTERFERENCE PROBLEM

A parallel development to the collocation technique is found in the Laplace transform solution of the sonic wind tunnel wall interference problem [Ref. 4].

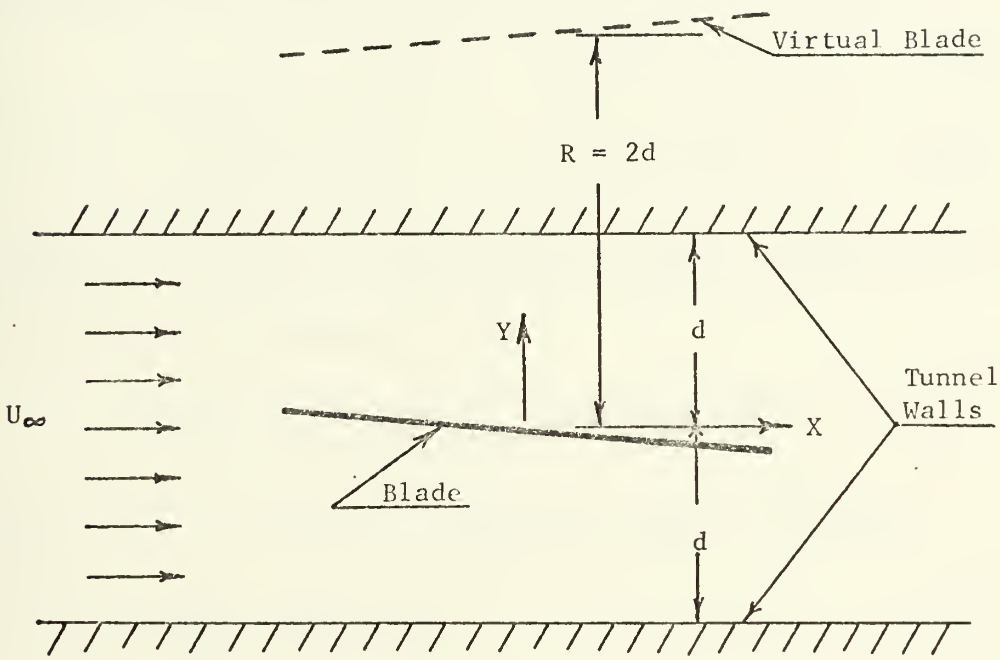


Figure 7. SONIC WIND TUNNEL

Figure 7 shows the arrangement of a blade facing a solid wall. Due to symmetry, the solution for the flow between the blade and the wall will be the same as that which would exist between the blade and the virtual blade beyond the wall. Hence it is the same solution expected of an unstaggered cascade with π phase shift between the blades.

The basic equation for oscillatory sonic flow from Section II is

$$\psi_{yy} - Q\psi_x + K^2\psi = 0 \ . \qquad \text{(II-49)}$$

The boundary-value problem can be formulated as follows:

1. There is no disturbance upstream of the blade leading edge.

2. Flow tangency conditions must be satisfied along the blade and along the solid wall.

This latter condition may be stated mathematically,

$$\psi_y(X,0) = -v(X) \qquad \text{at } Y = 0 \qquad \text{(III-48)}$$

and

$$\psi_y(X,d) = 0 \qquad \text{at } Y = \pm d \qquad \text{(III-49)}$$

Denoting the Laplace transformation of a function with respect to X by an overbar, i.e.,

$$\overline{f}(p) = \mathcal{L}\, f(X) = \int_0^{\infty} e^{-pX} f(X)\, dX \ ,$$

one obtains for the Laplace transform of Eqn. (II-49)

$$\overline{\psi}_{yy} - \lambda^2\overline{\psi} = 0 \qquad \text{(III-50)}$$

where

$$\lambda^2 = Qp - K^2 \qquad \text{(III-51)}$$

The boundary conditions transform into

$$\overline{\psi}_y = -\overline{v}(p) \qquad Y = 0 \qquad \text{(III-52)}$$

and

$$\overline{\psi}_y = 0 \qquad Y = \pm d \qquad \text{(III-53)}$$

A solution to (III-50) satisfying (III-52) and (III-53) has been given by Miles [Ref. 19],

$$\overline{\psi}(p,Y) = \frac{1}{\lambda}\, \overline{v}(p)\mathrm{CSCH}(\lambda d)\mathrm{COSH}[\lambda(d-|Y|)]\mathrm{sgn}(Y) \qquad \text{(III-54)}$$

Restricting oneself to the flow between the upper tunnel wall $Y = d$ and the blade one has $Y \geq 0$. Hence the absolute value sign may be dropped.

To invert (III-54), a suitable expansion of the equation was given by Miles as

$$\bar{\Psi}(p,Y) = \frac{1}{\lambda}\,\bar{v}(p) \sum_{N=0}^{\infty} \left(\mathrm{EXP}[-\lambda(2Nd+Y)] + \mathrm{EXP}\{-\lambda[(2N+2)d-Y]\}\right) \tag{III-55}$$

Recalling the Laplace transform of

$$f(X) = X^{-1/2}\,e^{-\alpha/4X} \qquad \mathrm{Re}\,\alpha \geq 0$$

being

$$\bar{f}(p) = \int_0^{\infty} e^{-pX} f(X)\,dX = \sqrt{\pi}\,\frac{e^{-\sqrt{\alpha p}}}{\sqrt{p}}\,, \quad \mathrm{Re}\ p > 0\,, \tag{III-56}$$

one has for

$$\bar{f}(p,Y) = \frac{1}{\lambda}\,\mathrm{EXP}[-\lambda(2Nd+Y)]$$

where $\lambda = \sqrt{Qp-K^2}$

the inversion

$$f(X,Y) = -\frac{\mathrm{EXP}\left[\frac{K^2}{Q}X - \frac{Q}{4}\frac{(2Nd+Y)^2}{X}\right]}{\sqrt{\pi Q X}}\;. \tag{III-57}$$

Therefore, with the aid of the convolution theorem, one obtains the following solution:

$$\psi(X,Y) = -\frac{1}{\sqrt{\pi Q}} \int_{-1}^{X} \frac{v(S)}{\sqrt{X-S}}\,\mathrm{EXP}\left[\frac{K^2}{Q}(X-S)\right] \cdot$$

$$\cdot \sum_{N=0}^{\infty} \left\{\mathrm{EXP}\left[-\frac{Q}{4}\frac{(2Nd+Y)^2}{X-S}\right] + \mathrm{EXP}\left[-\frac{Q}{4}\frac{[(2N+2)d+Y]^2}{X-S}\right]\right\} dS\;. \tag{III-58}$$

On the blade surface where Y = 0, this result simplifies to

$$\psi(X,0) = -\frac{1}{\sqrt{\pi Q}} \int_{-1}^{X} \frac{v(S)}{\sqrt{X-S}} \, \mathrm{EXP}[\frac{K^2}{Q}(X-S)] \cdot$$

$$\cdot \sum_{N=0}^{\infty} \left\{ \mathrm{EXP}[-\frac{QN^2d^2}{X-S}] + \mathrm{EXP}[-\frac{Q}{4}\frac{(2N+2)^2d^2}{X-S}] \right\} dS. \qquad \text{(III-59)}$$

For steady flow where K = 0, one obtains the further simplification,

$$\psi(X,0) = -\frac{1}{\sqrt{\pi\lambda}} \left\{ \int_{-1}^{X} \frac{v(S)}{\sqrt{X-S}} dS \right.$$

$$\left. + 2 \sum_{N=1}^{\infty} \int_{-1}^{X} \frac{v(S)}{\sqrt{X-S}} \, \mathrm{EXP}[-\lambda \frac{N^2d^2}{X-S}] \, dS \right\}. \qquad \text{(III-60)}$$

This expression was previously given by Sandeman [Ref. 4] in his analysis of choked wind tunnel flow. Equation (III-58) represents the generalization to oscillatory flow.

Appendix B-II describes the translation of (III-60) into a computer program. The solution of this function provides the steady state check of the collocation technique as is shown in Tables III and IV.

IV. PRESENTATION OF RESULTS

Table II presents the interference coefficients for the steady state case obtained for the controlling parameters indicated in the tables with the computer routine developed in Appendix A.

Table III contains the value of the perturbation potential along the reference blade obtained from the routine developed in Appendix B-I and utilizing the interference coefficients of Table II. These results may be compared with those in Table IV obtained from the steady state Laplace transform solution for the sonic wind tunnel wall interference problem of Section III-E and programmed in Appendix B-II.

Table V demonstrates the erroneous correction integrals obtained from Subroutine COR of Appendix A as the result of computer numerical inaccuracies. A comparison of the interference coefficients for two cases before and after the anomaly was detected is presented in Table VI.

A sample of the unsteady interference coefficients for the case $\lambda = 0$ is presented in Table VII.

Table VIII compares a sample steady state output with the output of the general case $\lambda \neq 0$, $K \neq 0$, for very small K.

TABLE II INTERFERENCE FUNCTION COEFFICIENTS

```
DL= 0.200          DK= 0.0              RHO= 1.000          AL= 0.0
DEL(00)=(-2.0453E 00, 0.0         )   DEL(10)=( 2.0453E 00, 0.0          )
DEL(01)=(-1.2504E 00, 0.0         )   DEL(11)=( 1.2504E 00, 0.0          )
DEL(02)=( 3.0619E-01, 0.0         )   DEL(12)=(-3.0619E-01, 0.0          )
DEL(03)=(-1.5100E-01, 0.0         )   DEL(13)=( 1.5100E-01, 0.0          )
DEL(04)=( 1.0639E-01, 0.0         )   DEL(14)=(-1.0639E-01, 0.0          )
DEL(05)=(-6.7890E-02, 0.0         )   DEL(15)=( 6.7890E-02, 0.0          )
DEL(06)=(-2.0573E-02, 0.0         )   DEL(16)=( 2.0573E-02, 0.0          )
DEL(07)=( 3.3798E-03, 0.0         )   DEL(17)=(-3.3798E-03, 0.0          )
DEL(08)=( 1.3068E-01, 0.0         )   DEL(18)=(-1.3068E-01, 0.0          )
DEL(09)=(-9.7604E-02, 0.0         )   DEL(19)=( 9.7604E-02, 0.0          )

DL= 0.200          DK= 0.0              RHO= 2.000          AL= 0.0
DEL(00)=(-8.0229E-01, 0.0         )   DEL(10)=( 8.0229E-01, 0.0          )
DEL(01)=(-6.1007E-01, 0.0         )   DEL(11)=( 6.1007E-01, 0.0          )
DEL(02)=( 1.4253E-01, 0.0         )   DEL(12)=(-1.4253E-01, 0.0          )
DEL(03)=(-6.6098E-02, 0.0         )   DEL(13)=( 6.6098E-02, 0.0          )
DEL(04)=( 2.7521E-02, 0.0         )   DEL(14)=(-2.7521E-02, 0.0          )
DEL(05)=(-1.2786E-02, 0.0         )   DEL(15)=( 1.2786E-02, 0.0          )
DEL(06)=( 5.6872E-02, 0.0         )   DEL(16)=(-5.6872E-02, 0.0          )
DEL(07)=(-5.7169E-02, 0.0         )   DEL(17)=( 5.7169E-02, 0.0          )
DEL(08)=(-4.0133E-02, 0.0         )   DEL(18)=( 4.0133E-02, 0.0          )
DEL(09)=( 5.0580E-02, 0.0         )   DEL(19)=(-5.0580E-02, 0.0          )
```

TABLE II INTERFERENCE FUNCTION COEFFICIENTS

```
DL= 0.200            DK= 0.0               RHO= 5.000          AL= 0.0
DEL(00)=(-1.1538E-01, 0.0        )    DEL(10)=( 1.1538E-01, 0.0        )
DEL(01)=(-1.8925E-01, 0.0        )    DEL(11)=( 1.8925E-01, 0.0        )
DEL(02)=( 7.1354E-03, 0.0        )    DEL(12)=(-7.1354E-03, 0.0        )
DEL(03)=( 2.2944E-02, 0.0        )    DEL(13)=(-2.2944E-02, 0.0        )
DEL(04)=(-2.5751E-02, 0.0        )    DEL(14)=( 2.5751E-02, 0.0        )
DEL(05)=( 2.8710E-02, 0.0        )    DEL(15)=(-2.8710E-02, 0.0        )
DEL(06)=(-6.2098E-02, 0.0        )    DEL(16)=( 6.2098E-02, 0.0        )
DEL(07)=( 4.1338E-02, 0.0        )    DEL(17)=(-4.1338E-02, 0.0        )
DEL(08)=( 5.2630E-02, 0.0        )    DEL(18)=(-5.2630E-02, 0.0        )
DEL(09)=(-5.1547E-02, 0.0        )    DEL(19)=( 5.1547E-02, 0.0        )

DL= 0.200            DK= 0.0               RHO=10.000          AL= 0.0
DEL(00)=(-1.5653E-03, 0.0        )    DEL(10)=( 1.5653E-03, 0.0        )
DEL(01)=(-8.5002E-03, 0.0        )    DEL(11)=( 8.5002E-03, 0.0        )
DEL(02)=(-1.4889E-02, 0.0        )    DEL(12)=( 1.4889E-02, 0.0        )
DEL(03)=(-5.3168E-03, 0.0        )    DEL(13)=( 5.3168E-03, 0.0        )
DEL(04)=( 7.0907E-03, 0.0        )    DEL(14)=(-7.0907E-03, 0.0        )
DEL(05)=(-6.0284E-04, 0.0        )    DEL(15)=( 6.0284E-04, 0.0        )
DEL(06)=(-4.9886E-03, 0.0        )    DEL(16)=( 4.9886E-03, 0.0        )
DEL(07)=( 3.8379E-03, 0.0        )    DEL(17)=(-3.8379E-03, 0.0        )
DEL(08)=( 1.6985E-03, 0.0        )    DEL(18)=(-1.6985E-03, 0.0        )
DEL(09)=(-2.1847E-03, 0.0        )    DEL(19)=( 2.1847E-03, 0.0        )
```

POTENTIAL SOLUTION FOR THE FOLLOWING PARAMETERS

DL= 0.200 DK= 0.0 RHO= 1.000

THE INTERFERENCE COEFFICIENTS

DEL(00)=-2.0453110E 00	DEL(10)= 2.0453110E 00
DEL(01)=-1.2504177E 00	DEL(11)= 1.2504177E 00
DEL(02)= 3.0618846E-01	DEL(12)=-3.0618846E-01
DEL(03)=-1.5100145E-01	DEL(13)= 1.5100145E-01
DEL(04)= 1.0638958E-01	DEL(14)=-1.0638958E-01
DEL(05)=-6.7890167E-02	DEL(15)= 6.7890167E-02
DEL(06)=-2.0572830E-02	DEL(16)= 2.0572830E-02
DEL(07)= 3.3797889E-03	DEL(17)=-3.3797889E-03
DEL(08)= 1.3067639E-01	DEL(18)=-1.3067639E-01
DEL(09)=-9.7604275E-02	DEL(19)= 9.7604275E-02

VALUE OF THE POTENTIAL ALONG THE BLADE

STATION	POTENTIAL
X=-0.90	POT= 1.1780910E 00
X=-0.80	POT= 2.1757832E 00
X=-0.70	POT= 3.1754112E 00
X=-0.60	POT= 4.1755638E 00
X=-0.50	POT= 5.1756020E 00
X=-0.40	POT= 6.1755772E 00
X=-0.30	POT= 7.1755638E 00
X=-0.20	POT= 8.1755724E 00
X=-0.10	POT= 9.1755791E 00
X=-0.00	POT= 1.0175587E 01
X= 0.10	POT= 1.1175572E 01
X= 0.20	POT= 1.2175570E 01
X= 0.30	POT= 1.3175570E 01
X= 0.40	POT= 1.4175561E 01
X= 0.50	POT= 1.5175563E 01
X= 0.60	POT= 1.6175568E 01
X= 0.70	POT= 1.7175552E 01
X= 0.80	POT= 1.8175476E 01
X= 0.90	POT= 1.9175308E 01
X= 1.00	POT= 2.0176529E 01

TABLE III

POTENTIAL SOLUTION FOR THE FOLLOWING PARAMETERS

DL= 0.200 DK= 0.0 RHO= 2.000

THE INTERFERENCE COEFFICIENTS

DEL(00)=-8.0229187E-01	DEL(10)= 8.0229187E-01
DEL(01)=-6.1007094E-01	DEL(11)= 6.1007094E-01
DEL(02)= 1.4253497E-01	DEL(12)=-1.4253497E-01
DEL(03)=-6.6098273E-02	DEL(13)= 6.6098273E-02
DEL(04)= 2.7520720E-02	DEL(14)=-2.7520720E-02
DEL(05)=-1.2785759E-02	DEL(15)= 1.2785759E-02
DEL(06)= 5.6872237E-02	DEL(16)=-5.6872237E-02
DEL(07)=-5.7169359E-02	DEL(17)= 5.7169359E-02
DEL(08)=-4.0132940E-02	DEL(18)= 4.0132940E-02
DEL(09)= 5.0579518E-02	DEL(19)=-5.0579518E-02

VALUE OF THE POTENTIAL ALONG THE BLADE

STATION	POTENTIAL
X=-0.90	POT= 8.1581414E-01
X=-0.80	POT= 1.3204842E 00
X=-0.70	POT= 1.8202248E 00
X=-0.60	POT= 2.3200750E 00
X=-0.50	POT= 2.8200855E 00
X=-0.40	POT= 3.3201075E 00
X=-0.30	POT= 3.8201075E 00
X=-0.20	POT= 4.3200874E 00
X=-0.10	POT= 4.8200750E 00
X=-0.00	POT= 5.3200960E 00
X= 0.10	POT= 5.8200798E 00
X= 0.20	POT= 6.3200665E 00
X= 0.30	POT= 6.8200712E 00
X= 0.40	POT= 7.3200731E 00
X= 0.50	POT= 7.8200769E 00
X= 0.60	POT= 8.3200636E 00
X= 0.70	POT= 8.8200598E 00
X= 0.80	POT= 9.3201199E 00
X= 0.90	POT= 9.8202019E 00
X= 1.00	POT= 1.0319364E 01

TABLE III

POTENTIAL SOLUTION FOR THE FOLLOWING PARAMETERS

DL= 0.200 DK= 0.0 RHO= 5.000

THE INTERFERENCE COEFFICIENTS

DEL(00)=-1.1538458E-01	DEL(10)= 1.1538458E-01
DEL(01)=-1.8925136E-01	DEL(11)= 1.8925136E-01
DEL(02)= 7.1354136E-03	DEL(12)=-7.1354136E-03
DEL(03)= 2.2944096E-02	DEL(13)=-2.2944096E-02
DEL(04)=-2.5751479E-02	DEL(14)= 2.5751479E-02
DEL(05)= 2.8709877E-02	DEL(15)=-2.8709877E-02
DEL(06)=-6.2098347E-02	DEL(16)= 6.2098347E-02
DEL(07)= 4.1337617E-02	DEL(17)=-4.1337617E-02
DEL(08)= 5.2630220E-02	DEL(18)=-5.2630220E-02
DEL(09)=-5.1547259E-02	DEL(19)= 5.1547259E-02

VALUE OF THE POTENTIAL ALONG THE BLADE

STATION	POTENTIAL
X=-0.90	POT= 7.9726553E-01
X=-0.80	POT= 1.1282368E 00
X=-0.70	POT= 1.3855448E 00
X=-0.60	POT= 1.6115170E 00
X=-0.50	POT= 1.8232851E 00
X=-0.40	POT= 2.0286140E 00
X=-0.30	POT= 2.2310429E 00
X=-0.20	POT= 2.4321480E 00
X=-0.10	POT= 2.6326466E 00
X=-0.00	POT= 2.8328724E 00
X= 0.10	POT= 3.0329742E 00
X= 0.20	POT= 3.2330236E 00
X= 0.30	POT= 3.4330444E 00
X= 0.40	POT= 3.6330500E 00
X= 0.50	POT= 3.8330574E 00
X= 0.60	POT= 4.0330648E 00
X= 0.70	POT= 4.2330627E 00
X= 0.80	POT= 4.4330502E 00
X= 0.90	POT= 4.6330080E 00
X= 1.00	POT= 4.8337231E 00

TABLE III

POTENTIAL SOLUTION FOR THE FOLLOWING PARAMETERS

DL= 0.200 DK= 0.0 RHO= 10.000

THE INTERFERENCE COEFFICIENTS

DEL(00)=-1.5652659E-03	DEL(10)= 1.5652659E-03
DEL(01)=-8.5002407E-03	DEL(11)= 8.5002407E-03
DEL(02)=-1.4888730E-02	DEL(12)= 1.4888730E-02
DEL(03)=-5.3168200E-03	DEL(13)= 5.3168200E-03
DEL(04)= 7.0907287E-03	DEL(14)=-7.0907287E-03
DEL(05)=-6.0283695E-04	DEL(15)= 6.0283695E-04
DEL(06)=-4.9886145E-03	DEL(16)= 4.9886145E-03
DEL(07)= 3.8378909E-03	DEL(17)=-3.8378909E-03
DEL(08)= 1.6985249E-03	DEL(18)=-1.6985249E-03
DEL(09)=-2.1847319E-03	DEL(19)= 2.1847319E-03

VALUE OF THE POTENTIAL ALONG THE BLADE

STATION	POTENTIAL
X=-0.90	POT= 7.9786545E-01
X=-0.80	POT= 1.1283703E 00
X=-0.70	POT= 1.3819685E 00
X=-0.60	POT= 1.5957613E 00
X=-0.50	POT= 1.7841234E 00
X=-0.40	POT= 1.9544506E 00
X=-0.30	POT= 2.1111937E 00
X=-0.20	POT= 2.2573242E 00
X=-0.10	POT= 2.3949909E 00
X=-0.00	POT= 2.5258188E 00
X= 0.10	POT= 2.6510839E 00
X= 0.20	POT= 2.7718019E 00
X= 0.30	POT= 2.8888006E 00
X= 0.40	POT= 3.0027514E 00
X= 0.50	POT= 3.1142006E 00
X= 0.60	POT= 3.2235994E 00
X= 0.70	POT= 3.3313122E 00
X= C.80	POT= 3.4376440E 00
X= 0.90	POT= 3.5428400E 00
X= 1.00	POT= 3.6471376E 00

TABLE III

TABLE IV VALUE OF THE POTENTIAL ALONG THE BLADE--WALL INTERFERENCE TECHNIQUE

SOLUTION FOR THE FOLLOWING PARAMETERS

DL= 0.200 DK= 0.0 RHO= 1.000

STATION		POTENTIAL
X=-0.90	POT=	1.16667E 00
X=-0.80	POT=	2.16667E 00
X=-0.70	POT=	3.16666E 00
X=-0.60	POT=	4.16665E 00
X=-0.50	POT=	5.16665E 00
X=-0.40	POT=	6.16664E 00
X=-0.30	POT=	7.16665E 00
X=-0.20	POT=	8.16664E 00
X=-0.10	POT=	9.16663E 00
X=-0.00	POT=	1.01666E 01
X= 0.10	POT=	1.11667E 01
X= 0.20	POT=	1.21666E 01
X= 0.30	POT=	1.31666E 01
X= 0.40	POT=	1.41666E 01
X= 0.50	POT=	1.51666E 01
X= 0.60	POT=	1.61666E 01
X= 0.70	POT=	1.71666E 01
X= 0.80	POT=	1.81666E 01
X= 0.90	POT=	1.91666E 01
X= 1.00	POT=	2.01666E 01

TABLE IV VALUE OF THE POTENTIAL ALONG THE BLADE--WALL INTERFERENCE TECHNIQUE

SOLUTION FOR THE FOLLOWING PARAMETERS

DL= 0.200 DK= 0.0 RHO= 2.000

STATION		POTENTIAL
X=-0.90	POT=	8.31876E-01
X=-C.80	POT=	1.33332E 00
X=-0.70	POT=	1.83333E 00
X=-0.60	POT=	2.33333E 00
X=-0.50	POT=	2.83333E 00
X=-0.40	POT=	3.33333E 00
X=-0.30	POT=	3.83333E 00
X=-0.20	POT=	4.33333E 00
X=-0.10	POT=	4.83333E 00
X=-0.00	POT=	5.33332E 00
X= 0.10	POT=	5.83333E 00
X= 0.20	POT=	6.33332E 00
X= 0.30	POT=	6.83331E 00
X= 0.40	POT=	7.33331E 00
X= 0.50	POT=	7.83332E 00
X= 0.60	POT=	8.33331E 00
X= 0.70	POT=	8.83331E 00
X= 0.80	POT=	9.33331E 00
X= 0.90	POT=	9.83332E 00
X= 1.00	POT=	1.03333E 01

TABLE IV VALUE OF THE POTENTIAL ALONG THE BLADE--WALL INTERFERENCE TECHNIQUE

SOLUTION FOR THE FOLLOWING PARAMETERS

DL= 0.200 DK= 0.0 RHO= 5.000

STATION	POTENTIAL	
X=-0.90	POT=	7.97885E-01
X=-0.80	POT=	1.12867E 00
X=-0.70	POT=	1.38590E 00
X=-0.60	POT=	1.61180E 00
X=-0.50	POT=	1.82356E 00
X=-0.40	POT=	2.02889E 00
X=-0.30	POT=	2.23132E 00
X=-0.20	POT=	2.43242E 00
X=-0.10	POT=	2.63291E 00
X=-0.00	POT=	2.83314E 00
X= 0.10	POT=	3.03325E 00
X= 0.20	POT=	3.23329E 00
X= 0.30	POT=	3.43331E 00
X= 0.40	POT=	3.63332E 00
X= 0.50	POT=	3.83333E 00
X= 0.60	POT=	4.03332E 00
X= 0.70	POT=	4.23333E 00
X= 0.80	POT=	4.43333E 00
X= 0.90	POT=	4.63333E 00
X= 1.00	POT=	4.83333E 00

TABLE IV VALUE OF THE POTENTIAL ALONG THE BLADE--WALL INTERFERENCE TECHNIQUE

SOLUTION FOR THE FOLLOWING PARAMETERS

DL= 0.200 DK= 0.0 RHO= 10.000

STATION	POTENTIAL	
X=-0.90	POT=	7.97885E-01
X=-0.80	POT=	1.12838E 00
X=-0.70	POT=	1.38198E 00
X=-0.60	POT=	1.59577E 00
X=-0.50	POT=	1.78413E 00
X=-0.40	POT=	1.95446E 00
X=-0.30	POT=	2.11120E 00
X=-0.20	POT=	2.25733E 00
X=-0.10	POT=	2.39500E 00
X=-0.00	POT=	2.52582E 00
X= 0.10	POT=	2.65109E 00
X= 0.20	POT=	2.77181E 00
X= 0.30	POT=	2.88880E 00
X= 0.40	POT=	3.00275E 00
X= 0.50	POT=	3.11420E 00
X= 0.60	POT=	3.22360E 00
X= 0.70	POT=	3.33132E 00
X= 0.80	POT=	3.43765E 00
X= 0.90	POT=	3.54285E 00
X= 1.00	POT=	3.64711E 00

SOLUTIONS FOR THE FOLLOWING PARAMETERS TABLE VA

DL= 0.0 DK= 0.500 RHO= 0.500

DL=0, CORRECTION USED

SIN AND COS FRESNEL INTEGRAL VALUE FOR UPPER LIMIT AR..

SSS= 0.38822D 00 CCC= 0.48815D 00 AR= 0.12500D 02

SOLUTION TO INTEGRAL(X**(J-2.5)SIN OR COS(E/X) ON LIMITS
ZERO TO AL= 5.000D-03 AND FOR E= 6.250D-02

J	SIN INTEGRAL	COS INTEGRAL
1	1.1208D 00	1.1885D-01
2	5.4775D-03	1.0112D-03
3	2.6500D-05	6.9534D-06
4	1.2694D-07	4.3037D-08
5	6.0104D-10	2.5309D-10
6	2.8637D-12	1.4515D-12
7	1.3830D-14	7.5455D-15
8	1.9751D-16	1.1803D-16
9	1.9863D-16	-3.2614D-16
10	4.5313D-15	2.7593D-15
11	-1.6338D-14	2.6829D-14
12	-6.1534D-15	-3.7471D-15
13	1.4279D-14	-2.3449D-14

SOLUTIONS FOR THE FOLLOWING PARAMETERS TABLE VB

DL= 0.0 DK= 0.500 RHO= 5.000

DL=0, CORRECTION USED

SIN AND COS FRESNEL INTEGRAL VALUE FOR UPPER LIMIT AR..

SSS= 0.47198D 00 CCC= 0.47791D 00 AR= 0.12500D 03

SOLUTION TO INTEGRAL(X**(J-2.5)SIN OR COS(E/X) ON LIMITS
ZERO TO AL= 5.000D-02 AND FOR E= 6.250D 00

J	SIN INTEGRAL	COS INTEGRAL
1	2.8093D-02	2.2152D-02
2	1.3946D-03	1.1193D-03
3	7.2128D-05	6.0492D-05
4	1.3478D-05	-4.1827D-06
5	-1.2389D-05	-1.7777D-05
6	-2.4882D-05	1.7451D-05
7	1.9823D-05	2.8285D-05
8	8.7655D-05	-6.1431D-05
9	-1.0363D-02	-1.4787D-02
10	2.0544D 01	-1.4398D 01
11	8.5248D 03	1.2164D 04
12	-2.7898D 05	1.9552D 05
13	-7.4507D 07	-1.0631D 08

INTERFERENCE FUNCTION COEFFICIENTS

```
DL= 0.0                 DK= 0.500           RHO= 0.500              AL= 0.0
DEL(00)=(-6.4001E-01, 1.9072E 00)      DEL(10)=( 6.4001E-01,-1.9072E 00)
DEL(01)=(-9.3527E-01, 7.7419E-01)      DEL(11)=( 9.3527E-01,-7.7419E-01)
DEL(02)=(-6.8607E-02,-4.4571E-01)      DEL(12)=( 6.8607E-02, 4.4571E-01)
DEL(03)=(-2.9892E-02, 1.8759E-01)      DEL(13)=( 2.9892E-02,-1.8759E-01)
DEL(04)=( 3.8507E-02,-1.2267E-01)      DEL(14)=(-3.8507E-02, 1.2267E-01)
DEL(05)=(-3.3817E-02, 6.7233E-02)      DEL(15)=( 3.3817E-02,-6.7233E-02)
DEL(06)=(-5.7924E-02, 3.1015E-02)      DEL(16)=( 5.7924E-02,-3.1015E-02)
DEL(07)=( 6.4039E-02, 1.6917E-02)      DEL(17)=(-6.4039E-02,-1.6917E-02)
DEL(08)=( 9.6658E-02,-1.7150E-01)      DEL(18)=(-9.6658E-02, 1.7150E-01)
DEL(09)=(-9.7952E-02, 1.1401E-01)      DEL(19)=( 9.7952E-02,-1.1401E-01)

DL= 0.0                 DK= 0.500           RHO= 0.500              AL= 0.00500
DEL(00)=(-6.4001E-01, 1.9072E 00)      DEL(10)=( 6.4001E-01,-1.9072E 00)
DEL(01)=(-9.3527E-01, 7.7419E-01)      DEL(11)=( 9.3527E-01,-7.7419E-01)
DEL(02)=(-6.8607E-02,-4.4571E-01)      DEL(12)=( 6.8607E-02, 4.4571E-01)
DEL(03)=(-2.9892E-02, 1.8759E-01)      DEL(13)=( 2.9892E-02,-1.8759E-01)
DEL(04)=( 3.8507E-02,-1.2267E-01)      DEL(14)=(-3.8507E-02, 1.2267E-01)
DEL(05)=(-3.3817E-02, 6.7235E-02)      DEL(15)=( 3.3817E-02,-6.7235E-02)
DEL(06)=(-5.7924E-02, 3.1017E-02)      DEL(16)=( 5.7924E-02,-3.1017E-02)
DEL(07)=( 6.4039E-02, 1.6914E-02)      DEL(17)=(-6.4039E-02,-1.6914E-02)
DEL(08)=( 9.6658E-02,-1.7150E-01)      DEL(18)=(-9.6658E-02, 1.7150E-01)
DEL(09)=(-9.7952E-02, 1.1402E-01)      DEL(19)=( 9.7952E-02,-1.1402E-01)
```

TABLE VIA INTERFERENCE COEFFICIENTS BEFORE AND AFTER ERROR DETECTION

INTERFERENCE FUNCTION COEFFICIENTS

```
DL= 0.0              DK= 0.500           RHO= 5.000           AL= 0.0
DEL(00)=( 9.8874E-01, 2.3480E-03)   DEL(10)=(-9.8874E-01,-2.3480E-03)
DEL(01)=(-9.3834E-03, 5.0453E-01)   DEL(11)=( 9.3834E-03,-5.0453E-01)
DEL(02)=(-2.5635E-03,-4.6483E-04)   DEL(12)=( 2.5635E-03, 4.6483E-04)
DEL(03)=(-1.3289E-03,-9.1403E-03)   DEL(13)=( 1.3289E-03, 9.1403E-03)
DEL(04)=( 1.3552E-02,-2.9720E-03)   DEL(14)=(-1.3552E-02, 2.9720E-03)
DEL(05)=( 4.2880E-03, 1.2744E-02)   DEL(15)=(-4.2880E-03,-1.2744E-02)
DEL(06)=(-1.4891E-02, 3.9460E-03)   DEL(16)=( 1.4891E-02,-3.9460E-03)
DEL(07)=(-2.4577E-03,-6.0711E-03)   DEL(17)=( 2.4577E-03, 6.0711E-03)
DEL(08)=( 6.2252E-03,-1.5752E-03)   DEL(18)=(-6.2252E-03, 1.5752E-03)
DEL(09)=(-3.6667E-06,-1.4728E-05)   DEL(19)=( 3.6667E-06, 1.4728E-05)

DL= 0.0              DK= 0.500           RHO= 5.000           AL= 0.05000
DEL(00)=( 1.8329E-02, 1.5829E-01)   DEL(10)=(-1.8329E-02,-1.5829E-01)
DEL(01)=(-9.8240E-01, 5.3604E-01)   DEL(11)=( 9.8240E-01,-5.3604E-01)
DEL(02)=(-2.2854E 00,-1.9778E 00)   DEL(12)=( 2.2854E 00, 1.9778E 00)
DEL(03)=( 6.3429E 00,-5.8245E 00)   DEL(13)=(-6.3429E 00, 5.8245E 00)
DEL(04)=( 7.3797E 00, 3.6977E 00)   DEL(14)=(-7.3797E 00,-3.6977E 00)
DEL(05)=(-1.9943E 01, 1.6795E 01)   DEL(15)=( 1.9943E 01,-1.6795E 01)
DEL(06)=(-6.6839E 00,-1.3534E 00)   DEL(16)=( 6.6839E 00, 1.3534E 00)
DEL(07)=( 2.9586E 01,-2.5003E 01)   DEL(17)=(-2.9586E 01, 2.5003E 01)
DEL(08)=( 1.3078E 00,-1.0888E 00)   DEL(18)=(-1.3078E 00, 1.0888E 00)
DEL(09)=(-1.5517E 01, 1.4587E 01)   DEL(19)=( 1.5517E 01,-1.4587E 01)
```

TABLE VIB   INTERFERENCE COEFFICIENTS BEFORE AND AFTER ERROR DETECTION

TABLE VII INTERFERENCE FUNCTION COEFFICIENTS

```
DL= 0.0              DK= 0.500              RHO= 0.500             AL= 0.00500
DEL(00)=(-6.4001E-01, 1.9072E 00)    DEL(10)=( 6.4001E-01,-1.9072E 00)
DEL(01)=(-9.3527E-01, 7.7419E-01)    DEL(11)=( 9.3527E-01,-7.7419E-01)
DEL(02)=(-6.8607E-02,-4.4571E-01)    DEL(12)=( 6.8607E-02, 4.4571E-01)
DEL(03)=(-2.9892E-02, 1.8759E-01)    DEL(13)=( 2.9892E-02,-1.8759E-01)
DEL(04)=( 3.8507E-02,-1.2267E-01)    DEL(14)=(-3.8507E-02, 1.2267E-01)
DEL(05)=(-3.3817E-02, 6.7235E-02)    DEL(15)=( 3.3817E-02,-6.7235E-02)
DEL(06)=(-5.7924E-02, 3.1017E-02)    DEL(16)=( 5.7924E-02,-3.1017E-02)
DEL(07)=( 6.4039E-02, 1.6914E-02)    DEL(17)=(-6.4039E-02,-1.6914E-02)
DEL(08)=( 9.6658E-02,-1.7150E-01)    DEL(18)=(-9.6658E-02, 1.7150E-01)
DEL(09)=(-9.7952E-02, 1.1402E-01)    DEL(19)=( 9.7952E-02,-1.1402E-01)

DL= 0.0              DK= 0.500              RHO= 1.000             AL= 0.01000
DEL(00)=(-9.1505E-02, 8.9399E-01)    DEL(10)=( 9.1505E-02,-8.9399E-01)
DEL(01)=(-4.2808E-01, 5.1503E-01)    DEL(11)=( 4.2808E-01,-5.1503E-01)
DEL(02)=(-5.8491E-02,-2.2545E-01)    DEL(12)=( 5.8491E-02, 2.2545E-01)
DEL(03)=( 1.7249E-02, 9.6792E-02)    DEL(13)=(-1.7249E-02,-9.6792E-02)
DEL(04)=( 3.7850E-02,-1.4023E-01)    DEL(14)=(-3.7850E-02, 1.4023E-01)
DEL(05)=(-1.2356E-01, 1.2207E-01)    DEL(15)=( 1.2356E-01,-1.2207E-01)
DEL(06)=(-1.4476E-01, 3.0627E-01)    DEL(16)=( 1.4476E-01,-3.0627E-01)
DEL(07)=( 3.6416E-01,-3.2108E-01)    DEL(17)=(-3.6416E-01, 3.2108E-01)
DEL(08)=( 9.2039E-02,-4.4756E-01)    DEL(18)=(-9.2039E-02, 4.4756E-01)
DEL(09)=(-2.5365E-01, 4.5904E-01)    DEL(19)=( 2.5365E-01,-4.5904E-01)
```

TABLE VII INTERFERENCE FUNCTION COEFFICIENTS

```
DL= 0.0               DK= 0.500              RHO= 1.500            AL= 0.01500
DEL(00)=( 4.6757E-02, 5.6529E-01)   DEL(10)=(-4.6757E-02,-5.6529E-01)
DEL(01)=(-2.2325E-01, 4.2943E-01)   DEL(11)=( 2.2325E-01,-4.2943E-01)
DEL(02)=(-1.5704E-01,-5.4023E-02)   DEL(12)=( 1.5704E-01, 5.4023E-02)
DEL(03)=( 3.5310E-01,-4.7883E-01)   DEL(13)=(-3.5310E-01, 4.7883E-01)
DEL(04)=( 7.3251E-01,-1.1851E 00)   DEL(14)=(-7.3251E-01, 1.1851E 00)
DEL(05)=(-2.3103E 00, 4.2034E 00)   DEL(15)=( 2.3103E 00,-4.2034E 00)
DEL(06)=(-1.8541E 00, 4.2023E 00)   DEL(16)=( 1.8541E 00,-4.2023E 00)
DEL(07)=( 5.2613E 00,-1.1150E 01)   DEL(17)=(-5.2613E 00, 1.1150E 01)
DEL(08)=( 8.2281E-01,-4.0267E 00)   DEL(18)=(-8.2281E-01, 4.0267E 00)
DEL(09)=(-3.0067E 00, 8.8394E 00)   DEL(19)=( 3.0067E 00,-8.8394E 00)

DL= 0.0               DK= 0.500              RHO= 2.000            AL= 0.02000
DEL(00)=( 1.8567E-01, 3.1111E-01)   DEL(10)=(-1.8567E-01,-3.1111E-01)
DEL(01)=( 1.4588E-01, 4.9891E-01)   DEL(11)=(-1.4588E-01,-4.9891E-01)
DEL(02)=(-4.3408E-01, 1.0499E 00)   DEL(12)=( 4.3408E-01,-1.0499E 00)
DEL(03)=(-4.2503E 00,-1.8332E 00)   DEL(13)=( 4.2503E 00, 1.8332E 00)
DEL(04)=( 1.8235E 00,-7.3036E 00)   DEL(14)=(-1.8235E 00, 7.3036E 00)
DEL(05)=( 1.8382E 01, 8.3875E 00)   DEL(15)=(-1.8382E 01,-8.3875E 00)
DEL(06)=(-2.7393E 00, 1.3482E 01)   DEL(16)=( 2.7393E 00,-1.3482E 01)
DEL(07)=(-3.0227E 01,-1.3183E 01)   DEL(17)=( 3.0227E 01, 1.3183E 01)
DEL(08)=( 1.2130E 00,-7.7521E 00)   DEL(18)=(-1.2130E 00, 7.7521E 00)
DEL(09)=( 1.6760E 01, 6.9288E 00)   DEL(19)=(-1.6760E 01,-6.9288E 00)
```

TABLE VII INTERFERENCE FUNCTION COEFFICIENTS

DL= 0.0 DK= 0.500	RHO= 5.000 AL= 0.05000
DEL(00)=(1.8329E-02, 1.5829E-01)	DEL(10)=(-1.8329E-02,-1.5829E-01)
DEL(01)=(-9.8240E-01, 5.3604E-01)	DEL(11)=(9.8240E-01,-5.3604E-01)
DEL(02)=(-2.2854E 00,-1.9778E 00)	DEL(12)=(2.2854E 00, 1.9778E 00)
DEL(03)=(6.3429E 00,-5.8245E 00)	DEL(13)=(-6.3429E 00, 5.8245E 00)
DEL(04)=(7.3797E 00, 3.6977E 00)	DEL(14)=(-7.3797E 00,-3.6977E 00)
DEL(05)=(-1.9943E 01, 1.6795E 01)	DEL(15)=(1.9943E 01,-1.6795E 01)
DEL(06)=(-6.6839E 00,-1.3534E 00)	DEL(16)=(6.6839E 00, 1.3534E 00)
DEL(07)=(2.9586E 01,-2.5003E 01)	DEL(17)=(-2.9586E 01, 2.5003E 01)
DEL(08)=(1.3078E 00,-1.0888E 00)	DEL(18)=(-1.3078E 00, 1.0888E 00)
DEL(09)=(-1.5517E 01, 1.4587E 01)	DEL(19)=(1.5517E 01,-1.4587E 01)

DL= 0.0 DK= 0.500	RHO=10.000 AL= 0.10000
DEL(00)=(-4.0697E-02, 1.5915E-02)	DEL(10)=(4.0697E-02,-1.5915E-02)
DEL(01)=(-9.2882E-02,-6.1628E-03)	DEL(11)=(9.2882E-02, 6.1628E-03)
DEL(02)=(2.3159E 00,-1.3794E-01)	DEL(12)=(-2.3159E 00, 1.3794E-01)
DEL(03)=(3.3040E 00, 4.5079E-01)	DEL(13)=(-3.3040E 00,-4.5079E-01)
DEL(04)=(-1.8222E 01, 2.4402E-01)	DEL(14)=(1.8222E 01,-2.4402E-01)
DEL(05)=(-2.4621E 01, 8.6918E-01)	DEL(15)=(2.4621E 01,-8.6918E-01)
DEL(06)=(4.1382E 01,-6.6481E-01)	DEL(16)=(-4.1382E 01, 6.6481E-01)
DEL(07)=(5.6122E 01,-8.9252E 00)	DEL(17)=(-5.6122E 01, 8.9252E 00)
DEL(08)=(-2.7337E 01, 4.6686E-01)	DEL(18)=(2.7337E 01,-4.6686E-01)
DEL(09)=(-3.7490E 01, 8.8355E 00)	DEL(19)=(3.7490E 01,-8.8355E 00)

TABLE VII INTERFERENCE FUNCTION COEFFICIENTS

```
DL= 0.0            DK= 1.000              RHO= 0.500            AL= 0.00500
DEL(00)=( 5.6994E-02, 1.4490E 00)   DEL(10)=(-5.6994E-02,-1.4490E 00)
DEL(01)=(-6.9615E-01, 5.6006E-01)   DEL(11)=( 6.9615E-01,-5.6006E-01)
DEL(02)=(-2.8831E-01,-4.8690E-01)   DEL(12)=( 2.8831E-01, 4.8690E-01)
DEL(03)=( 2.7632E-02, 2.1122E-01)   DEL(13)=(-2.7632E-02,-2.1122E-01)
DEL(04)=(-1.5099E-03,-1.5008E-01)   DEL(14)=( 1.5099E-03, 1.5008E-01)
DEL(05)=(-3.1021E-02, 9.5720E-02)   DEL(15)=( 3.1021E-02,-9.5720E-02)
DEL(06)=(-5.8784E-02, 1.3836E-01)   DEL(16)=( 5.8784E-02,-1.3836E-01)
DEL(07)=( 1.3799E-01,-9.9258E-02)   DEL(17)=(-1.3799E-01, 9.9258E-02)
DEL(08)=( 1.7517E-02,-3.0274E-01)   DEL(18)=(-1.7517E-02, 3.0274E-01)
DEL(09)=(-8.1509E-02, 2.5667E-01)   DEL(19)=( 8.1509E-02,-2.5667E-01)

DL= 0.0            DK= 1.000              RHO= 1.000            AL= 0.01000
DEL(00)=( 2.4261E-01, 6.5696E-01)   DEL(10)=(-2.4261E-01,-6.5696E-01)
DEL(01)=(-2.2522E-01, 5.0475E-01)   DEL(11)=( 2.2522E-01,-5.0475E-01)
DEL(02)=(-2.2097E-01,-1.5224E-01)   DEL(12)=( 2.2097E-01, 1.5224E-01)
DEL(03)=( 1.6615E-01,-2.4715E-01)   DEL(13)=(-1.6615E-01, 2.4715E-01)
DEL(04)=( 1.3848E-01,-9.5467E-01)   DEL(14)=(-1.3848E-01, 9.5467E-01)
DEL(05)=(-7.1255E-01, 2.7993E 00)   DEL(15)=( 7.1255E-01,-2.7993E 00)
DEL(06)=(-1.9721E-01, 3.2382E 00)   DEL(16)=( 1.9721E-01,-3.2382E 00)
DEL(07)=( 1.4099E 00,-7.6124E 00)   DEL(17)=(-1.4099E 00, 7.6124E 00)
DEL(08)=(-4.5689E-01,-3.0672E 00)   DEL(18)=( 4.5689E-01, 3.0672E 00)
DEL(09)=(-2.9057E-01, 6.1414E 00)   DEL(19)=( 2.9057E-01,-6.1414E 00)
```

TABLE VII INTERFERENCE FUNCTION COEFFICIENTS

DL= 0.0	DK= 1.000	RHO= 1.500	AL= 0.01500

DEL(00)=(4.5000E-01, 8.4206E-02)	DEL(10)=(-4.5000E-01,-8.4206E-02)
DEL(01)=(1.8850E 00, 1.7012E 00)	DEL(11)=(-1.8850E 00,-1.7012E 00)
DEL(02)=(-4.6762E 00, 7.3129E 00)	DEL(12)=(4.6762E 00,-7.3129E 00)
DEL(03)=(-2.1822E 01,-1.4191E 01)	DEL(13)=(2.1822E 01, 1.4191E 01)
DEL(04)=(2.1326E 01,-3.7423E 01)	DEL(14)=(-2.1326E 01, 3.7423E 01)
DEL(05)=(7.3379E 01, 4.7626E 01)	DEL(15)=(-7.3379E 01,-4.7626E 01)
DEL(06)=(-3.2624E 01, 6.0380E 01)	DEL(16)=(3.2624E 01,-6.0380E 01)
DEL(07)=(-1.0077E 02,-6.3109E 01)	DEL(17)=(1.0077E 02, 6.3109E 01)
DEL(08)=(1.5654E 01,-3.1274E 01)	DEL(18)=(-1.5654E 01, 3.1274E 01)
DEL(09)=(5.0221E 01, 2.9707E 01)	DEL(19)=(-5.0221E 01,-2.9707E 01)

DL= 0.0	DK= 1.000	RHO= 2.000	AL= 0.02000

DEL(00)=(7.2210E-01,-4.3992E-01)	DEL(10)=(-7.2210E-01, 4.3992E-01)
DEL(01)=(2.3186E 00, 1.8141E 00)	DEL(11)=(-2.3186E 00,-1.8141E 00)
DEL(02)=(-3.1775E 00, 5.0090E 00)	DEL(12)=(3.1775E 00,-5.0090E 00)
DEL(03)=(-7.2052E 00,-4.6812E 00)	DEL(13)=(7.2052E 00, 4.6812E 00)
DEL(04)=(4.7306E 00,-9.2184E 00)	DEL(14)=(-4.7306E 00, 9.2184E 00)
DEL(05)=(6.3154E 00, 6.8983E 00)	DEL(15)=(-6.3154E 00,-6.8983E 00)
DEL(06)=(-1.6314E 00, 7.9922E 00)	DEL(16)=(1.6314E 00,-7.9922E 00)
DEL(07)=(-2.5502E 00,-8.6041E 00)	DEL(17)=(2.5502E 00, 8.6041E 00)
DEL(08)=(-1.0280E 00,-3.4320E 00)	DEL(18)=(1.0280E 00, 3.4320E 00)
DEL(09)=(1.2822E 00, 5.1690E 00)	DEL(19)=(-1.2822E 00,-5.1690E 00)

TABLE VII INTERFERENCE FUNCTION COEFFICIENTS

```
DL= 0.0              DK= 1.500              RHO= 0.500             AL= 0.00500
DEL(00)=( 4.5391E-01, 1.1817E 00)   DEL(10)=(-4.5391E-01,-1.1817E 00)
DEL(01)=(-5.8275E-01, 4.3809E-01)   DEL(11)=( 5.8275E-01,-4.3809E-01)
DEL(02)=(-4.9712E-01,-5.2018E-01)   DEL(12)=( 4.9712E-01, 5.2018E-01)
DEL(03)=( 6.2010E-02, 2.5113E-01)   DEL(13)=(-6.2010E-02,-2.5113E-01)
DEL(04)=(-3.4702E-02,-2.0097E-01)   DEL(14)=( 3.4702E-02, 2.0097E-01)
DEL(05)=(-4.2637E-02, 1.7768E-01)   DEL(15)=( 4.2637E-02,-1.7768E-01)
DEL(06)=(-2.8836E-03, 3.3560E-01)   DEL(16)=( 2.8836E-03,-3.3560E-01)
DEL(07)=( 1.9561E-01,-3.8605E-01)   DEL(17)=(-1.9561E-01, 3.8605E-01)
DEL(08)=(-1.3230E-01,-5.1511E-01)   DEL(18)=( 1.3230E-01, 5.1511E-01)
DEL(09)=(-1.6452E-02, 5.4211E-01)   DEL(19)=( 1.6452E-02,-5.4211E-01)

DL= 0.0              DK= 1.500              RHO= 1.000             AL= 0.01000
DEL(00)=( 5.9815E-01, 2.9476E-01)   DEL(10)=(-5.9815E-01,-2.9476E-01)
DEL(01)=( 1.3168E-01, 3.0835E-01)   DEL(11)=(-1.3168E-01,-3.0835E-01)
DEL(02)=( 1.3317E 00, 7.7680E-01)   DEL(12)=(-1.3317E 00,-7.7680E-01)
DEL(03)=(-9.1215E 00, 4.7352E 00)   DEL(13)=( 9.1215E 00,-4.7352E 00)
DEL(04)=(-1.4289E 01,-6.6618E 00)   DEL(14)=( 1.4289E 01, 6.6618E 00)
DEL(05)=( 5.5213E 01,-2.5854E 01)   DEL(15)=(-5.5213E 01, 2.5854E 01)
DEL(06)=( 3.6534E 01, 8.7684E 00)   DEL(16)=(-3.6534E 01,-8.7684E 00)
DEL(07)=(-1.1478E 02, 5.7177E 01)   DEL(17)=( 1.1478E 02,-5.7177E 01)
DEL(08)=(-2.5142E 01,-1.2331E 00)   DEL(18)=( 2.5142E 01, 1.2331E 00)
DEL(09)=( 7.2441E 01,-4.0719E 01)   DEL(19)=(-7.2441E 01, 4.0719E 01)
```

TABLE VII INTERFERENCE FUNCTION COEFFICIENTS

```
DL= 0.0              DK= 1.500              RHO= 1.500              AL= 0.01500
DEL(00)=( 1.0872E 00,-2.4305E-01)   DEL(10)=(-1.0872E 00, 2.4305E-01)
DEL(01)=( 1.6653E 00, 3.3827E 00)   DEL(11)=(-1.6653E 00,-3.3827E 00)
DEL(02)=(-7.5439E 00, 3.8415E 00)   DEL(12)=( 7.5439E 00,-3.8415E 00)
DEL(03)=(-6.4762E 00,-1.2344E 01)   DEL(13)=( 6.4762E 00, 1.2344E 01)
DEL(04)=( 1.5659E 01,-8.9169E 00)   DEL(14)=(-1.5659E 01, 8.9169E 00)
DEL(05)=( 7.8549E 00, 1.6617E 01)   DEL(15)=(-7.8549E 00,-1.6617E 01)
DEL(06)=(-1.2695E 01, 8.7519E 00)   DEL(16)=( 1.2695E 01,-8.7519E 00)
DEL(07)=(-3.3243E 00,-1.1969E 01)   DEL(17)=( 3.3243E 00, 1.1969E 01)
DEL(08)=( 3.4906E 00,-4.1086E 00)   DEL(18)=(-3.4906E 00, 4.1086E 00)
DEL(09)=( 7.1226E-01, 4.9689E 00)   DEL(19)=(-7.1226E-01,-4.9689E 00)

DL= 0.0              DK= 1.500              RHO= 2.000              AL= 0.02000
DEL(00)=(-1.9828E-01,-3.6613E-01)   DEL(10)=( 1.9828E-01, 3.6613E-01)
DEL(01)=( 4.4313E-01,-1.0575E 00)   DEL(11)=(-4.4313E-01, 1.0575E 00)
DEL(02)=( 1.5029E 00, 3.7636E-01)   DEL(12)=(-1.5029E 00,-3.7636E-01)
DEL(03)=( 1.8064E 00,-3.9800E 00)   DEL(13)=(-1.8064E 00, 3.9800E 00)
DEL(04)=( 1.4026E 00,-2.4766E 00)   DEL(14)=(-1.4026E 00, 2.4766E 00)
DEL(05)=(-7.0044E 00, 3.5492E 01)   DEL(15)=( 7.0044E 00,-3.5492E 01)
DEL(06)=(-2.8965E 00, 4.4448E 00)   DEL(16)=( 2.8965E 00,-4.4448E 00)
DEL(07)=( 7.4004E 00,-7.1110E 01)   DEL(17)=(-7.4004E 00, 7.1110E 01)
DEL(08)=( 3.8101E-01,-2.3103E 00)   DEL(18)=(-3.8101E-01, 2.3103E 00)
DEL(09)=(-1.2210E 00, 4.3493E 01)   DEL(19)=( 1.2210E 00,-4.3493E 01)
```

TABLE VII INTERFERENCE FUNCTION COEFFICIENTS

```
DL= 0.0              DK= 2.000              RHO= 0.500             AL= 0.00500
DEL(00)=( 7.3954E-01, 9.5053E-01)   DEL(10)=(-7.3954E-01,-9.5053E-01)
DEL(01)=(-5.3530E-01, 3.2850E-01)   DEL(11)=( 5.3530E-01,-3.2850E-01)
DEL(02)=(-7.3352E-01,-5.1747E-01)   DEL(12)=( 7.3352E-01, 5.1747E-01)
DEL(03)=( 9.2360E-02, 3.0320E-01)   DEL(13)=(-9.2360E-02,-3.0320E-01)
DEL(04)=(-7.6259E-02,-2.8962E-01)   DEL(14)=( 7.6259E-02, 2.8962E-01)
DEL(05)=(-4.7236E-02, 3.7243E-01)   DEL(15)=( 4.7236E-02,-3.7243E-01)
DEL(06)=( 1.6200E-01, 6.8334E-01)   DEL(16)=(-1.6200E-01,-6.8334E-01)
DEL(07)=( 1.6875E-01,-1.0155E 00)   DEL(17)=(-1.6875E-01, 1.0155E 00)
DEL(08)=(-4.1890E-01,-8.5082E-01)   DEL(18)=( 4.1890E-01, 8.5082E-01)
DEL(09)=( 1.7750E-01, 1.0929E 00)   DEL(19)=(-1.7750E-01,-1.0929E 00)

DL= 0.0              DK= 2.000              RHO= 1.000             AL= 0.01000
DEL(00)=( 5.8019E-01, 1.0374E-01)   DEL(10)=(-5.8019E-01,-1.0374E-01)
DEL(01)=( 1.1588E 00, 7.4301E-02)   DEL(11)=(-1.1588E 00,-7.4301E-02)
DEL(02)=( 1.7872E 00, 4.1418E 00)   DEL(12)=(-1.7872E 00,-4.1418E 00)
DEL(03)=(-1.4714E 01, 6.5528E 00)   DEL(13)=( 1.4714E 01,-6.5528E 00)
DEL(04)=(-1.3141E 01,-2.3372E 01)   DEL(14)=( 1.3141E 01, 2.3372E 01)
DEL(05)=( 5.6242E 01,-2.3700E 01)   DEL(15)=(-5.6242E 01, 2.3700E 01)
DEL(06)=( 2.3682E 01, 3.9066E 01)   DEL(16)=(-2.3682E 01,-3.9066E 01)
DEL(07)=(-8.3080E 01, 3.6331E 01)   DEL(17)=( 8.3080E 01,-3.6331E 01)
DEL(08)=(-1.3432E 01,-2.0678E 01)   DEL(18)=( 1.3432E 01, 2.0678E 01)
DEL(09)=( 4.2600E 01,-1.9546E 01)   DEL(19)=(-4.2600E 01, 1.9546E 01)
```

TABLE VII INTERFERENCE FUNCTION COEFFICIENTS

```
DL= 0.0               DK= 2.000              RHO= 1.500            AL= 0.01500
DEL(00)=(-8.9812E-03,-9.9396E-01)   DEL(10)=( 8.9812E-03, 9.9396E-01)
DEL(01)=( 2.7173E 00,-1.6050E 00)   DEL(11)=(-2.7173E 00, 1.6050E 00)
DEL(02)=( 3.4268E 00, 4.0791E 00)   DEL(12)=(-3.4268E 00,-4.0791E 00)
DEL(03)=(-4.0185E 00, 3.5840E 00)   DEL(13)=( 4.0185E 00,-3.5840E 00)
DEL(04)=(-4.0352E 00,-5.5826E 00)   DEL(14)=( 4.0352E 00, 5.5826E 00)
DEL(05)=( 1.0605E 00, 3.9181E 00)   DEL(15)=(-1.0605E 00,-3.9181E 00)
DEL(06)=( 3.5970E 00, 6.1062E 00)   DEL(16)=(-3.5970E 00,-6.1062E 00)
DEL(07)=( 3.4634E-01,-1.4446E 01)   DEL(17)=(-3.4634E-01, 1.4446E 01)
DEL(08)=(-2.9542E 00,-3.6515E 00)   DEL(18)=( 2.9542E 00, 3.6515E 00)
DEL(09)=( 1.0824E 00, 1.0300E 01)   DEL(19)=(-1.0824E 00,-1.0300E 01)

DL= 0.0               DK= 2.000              RHO= 2.000            AL= 0.02000
DEL(00)=(-7.0111E-01, 2.0849E-01)   DEL(10)=( 7.0111E-01,-2.0849E-01)
DEL(01)=(-1.7059E 00,-3.3996E 00)   DEL(11)=( 1.7059E 00, 3.3996E 00)
DEL(02)=( 9.7961E 00,-3.2129E 00)   DEL(12)=(-9.7961E 00, 3.2129E 00)
DEL(03)=( 1.1316E 01, 3.2305E 01)   DEL(13)=(-1.1316E 01,-3.2305E 01)
DEL(04)=(-4.8702E 01, 6.2683E 00)   DEL(14)=( 4.8702E 01,-6.2683E 00)
DEL(05)=(-5.0269E 01,-1.0972E 02)   DEL(15)=( 5.0269E 01, 1.0972E 02)
DEL(06)=( 8.6721E 01,-2.7566E 00)   DEL(16)=(-8.6721E 01, 2.7566E 00)
DEL(07)=( 8.6941E 01, 1.4372E 02)   DEL(17)=(-8.6941E 01,-1.4372E 02)
DEL(08)=(-4.9922E 01,-2.5936E 00)   DEL(18)=( 4.9922E 01, 2.5936E 00)
DEL(09)=(-4.8683E 01,-6.4879E 01)   DEL(19)=( 4.8683E 01, 6.4879E 01)
```

TABLE VIII INTERFERENCE FUNCTION COEFFICIENTS

```
DL= 0.200              DK= 0.0                   RHO= 1.000             AL= 0.0
DEL(00)=(-2.0453E 00, 0.0         )   DEL(10)=( 2.0453E 00, 0.0         )
DEL(01)=(-1.2504E 00, 0.0         )   DEL(11)=( 1.2504E 00, 0.0         )
DEL(02)=( 3.0619E-01, 0.0         )   DEL(12)=(-3.0619E-01, 0.0         )
DEL(03)=(-1.5100E-01, 0.0         )   DEL(13)=( 1.5100E-01, 0.0         )
DEL(04)=( 1.0639E-01, 0.0         )   DEL(14)=(-1.0639E-01, 0.0         )
DEL(05)=(-6.7890E-02, 0.0         )   DEL(15)=( 6.7890E-02, 0.0         )
DEL(06)=(-2.0573E-02, 0.0         )   DEL(16)=( 2.0573E-02, 0.0         )
DEL(07)=( 3.3798E-03, 0.0         )   DEL(17)=(-3.3798E-03, 0.0         )
DEL(08)=( 1.3068E-01, 0.0         )   DEL(18)=(-1.3068E-01, 0.0         )
DEL(09)=(-9.7604E-02, 0.0         )   DEL(19)=( 9.7604E-02, 0.0         )

DL= 0.200              DK= 0.000                 RHO= 1.000             AL= 0.0
DEL(00)=(-2.0453E 00, 1.3414E-07)   DEL(10)=( 2.0453E 00,-1.3414E-07)
DEL(01)=(-1.2504E 00, 5.5256E-08)   DEL(11)=( 1.2504E 00,-5.5256E-08)
DEL(02)=( 3.0619E-01,-2.5136E-08)   DEL(12)=(-3.0619E-01, 2.5136E-08)
DEL(03)=(-1.5100E-01, 1.0586E-08)   DEL(13)=( 1.5100E-01,-1.0586E-08)
DEL(04)=( 1.0639E-01,-7.2418E-09)   DEL(14)=(-1.0639E-01, 7.2418E-09)
DEL(05)=(-6.7890E-02, 4.1421E-09)   DEL(15)=( 6.7890E-02,-4.1421E-09)
DEL(06)=(-2.0573E-02, 1.5717E-09)   DEL(16)=( 2.0573E-02,-1.5717E-09)
DEL(07)=( 3.3798E-03, 8.4642E-10)   DEL(17)=(-3.3798E-03,-8.4642E-10)
DEL(08)=( 1.3068E-01,-9.7137E-09)   DEL(18)=(-1.3068E-01, 9.7137E-09)
DEL(09)=(-9.7604E-02, 6.5800E-09)   DEL(19)=( 9.7604E-02,-6.5800E-09)
```

V. DISCUSSION OF RESULTS

A computer program was designed to compute interference coefficients for the collocation technique developed in Section III-A. For the steady case, the resulting coefficients in Table II were applied to the technique developed in Section III-C and III-D to provide the values of the blade surface potentials in Table III. For this same case, the sonic wind tunnel wall interference solution of Section III-E provided the data presented in Table IV which verified the collocation technique for that case.

Although no "separate source" checks were made for the coefficients presented in Table VII (The case of simple oscillation about the mid-chord with no translation.), every effort was made to verify the procedure. Table VIII compares the proven steady state solution with a general small K solution. Unsteady integral approximation techniques were justified in Sections III-B-1, -2, and -3 as summarized below.

Difficulties arising from the solution of the occurring integrals in the unsteady solution were not insurmountable but required careful handling. Whenever numerical integration of functions involving $\frac{1}{X}$ or $\frac{\text{SIN}}{\text{COS}}\left(\frac{1}{X}\right)$ near $X = 0$ is attempted, accuracy is suspect. One must closely inspect arguments, limits, and results to ensure validity. For example, for the unsteady case and $\lambda \neq 0$, with no apparent

means of computing the integral in the immediate vicinity of the origin, one must decide, against the shortcomings in computer capacity and programming methods, just what approximation must be applied and, once applied, how accurate it is. In this case, the question arose as to how closely one might approach the origin with numerical integration techniques before accuracy was degraded. At the same time the required computer time increased dramatically as the integrand became more ill-behaved.

The most difficult question that arose in this investigation was the accuracy of the correction integral in the case $\lambda = 0$. Once this approach was found valid for $J = 1$ through the calculations shown in Section III-B-2, there remained the need to improve the accuracy of the technique and ensure its validity for $J > 1$. In question were the number of terms needed in the truncated series shown in Equations (III-40) and (III-41), the validity of the integral reduction formula used in subroutine RED, and the selection of the upper limit of the correction integral.

Although four terms were used in the truncated series for the case $J = 1$ (Refer to Appendix A-IV), only three terms affected the significant figures of the results for the parameters used in this investigation. For $J = 2$, 3, and 4, the number of terms used was decreased by one for each computation. The primary motivation for this step was the inaccuracy of the integral reduction formula for large J and large R.

Referring to Tables VA and VB, it was determined that numerical inaccuracies began to occur for large J in the integral reduction routine. The effect of this inaccuracy was not important for small R but for the case of R = 5, for example, it overwhelmed the results. For integrals of the form

$$\int_0^{AL} U^{J-\frac{5}{2}} \begin{matrix} SIN \\ COS \end{matrix} (E/U)dU,$$

as J becomes large and for AL << 1, the value of the integral must become small. However, in the case of R = 0.5, the magnitude of the integral values diminished with increasing J up to J = 9 but increased for J > 9. In the case of R = 5, the anomaly in the computation affected integrals for J's as low as five. (In this latter case, it is obvious that the erroneous answer for J = 13 must affect the results.) The cause of this error was determined to be differences taken between large numbers in the reduction formula. The value obtained for the Fresnel integrals was only as accurate as the curve fit used in Subroutine CS. This error and the error generated in computing the SIN and COS functions occurring in the formula resulted in erroneous answers of a magnitude that was directly related to J and R. To correct the problem, the use of the integral reduction routine for $J \geq 5$ was eliminated.

For $J \geq 5$, numerical techniques were resorted to. Here, for AL << 1, the amplitude of the integrand was small

permitting one to neglect that portion of the integral near zero. This technique seemed to produce reasonable results, at least the results did not have large magnitudes that would have affected the final answers. The resulting integral values were in some cases less than the convergence limits placed on the numerical integration subroutine, ZSUM, and were, therefore, assumed to contain error not greater than the said limits. Further, as these integral values became small, it should have been possible to neglect them altogether since the values of those integrals with small (J) predominated in the series.

Comparative results for the case K = 0.5 and R = 0.5 and 5.0 are presented before and after the error was detected in Tables VIA and VIB.

The upper limit of the correction integral was selected such that the approximation $\frac{SIN}{COS}(E/U) = \frac{SIN}{COS}(DU + E/U)$ is valid in the range $0 \leq U \leq AL$. The point AL also needed to be within the capabilities of the numerical integration scheme on the limits $AL \leq U \leq X + 1$. Such a point was picked at AL = R/100 which ensures that (For $U \leq AL$) E/U was at least 1.0E4 larger than DU. Such an AL was also within the capabilities of the numerical technique over the range of parameters used in this investigation. Table IB demonstrates that the use of the truncated series in this approximation decreases its sensitivity to AL.

Computer time could have been reduced with a larger AL if one were willing to accept reasonable inaccuracies as

the term DU became more significant. A smaller AL increased computer time and the accuracy of the numerical integration technique became more and more questionable as the integral became more ill-behaved. Computer time for a single run of the program (for $\lambda = 0$) on the IBM 360/67 computer system was on the order of 15 minutes.

It should be noted that in the absence of a good error measurement yardstick, statements concerning more or less error are indeed relative. It is difficult to assess where computer numerical error or approximation error becomes more important. It is for this reason that an independent check of these results against another source is necessary. The Laplace transform solution for the sonic unsteady wind tunnel wall interference problem would, in theory, be such a check; however, this technique will result in integrals of the same form as those used in the collocation solution and will require similar (or the same) solution techniques. It will be difficult, then, to obtain a completely independent check of these results.

Once verification or modification of results presented in this investigation is in hand, application of Spreiter's local linearization technique alluded to in Section II should be applicable and lead to reasonable analytic results for transonic pressure and moment distribution prediction.

VI. SUMMARY

Following the basic approach offered by Gorelov for subsonic and low supersonic flows, a collocation technique was applied to the purely transonic case superimposing the isolated foil solution provided by Teipel with unknown interference potentials. A computer routine was developed which would generate the coefficients for the unknown interference source distributions in the steady case. The interference coefficients in hand, the potential along the blade was determined through a second routine. These results were favorably compared with the potential solution derived from a routine based on the Laplace transform treatment of the sonic wind tunnel wall interference problem for the steady case. All three of these computer routines initially utilized series expansions which reduced the occurring integrals to functions of the well known error function. Confirmation of these results led to an extension of the collocation approach into the unsteady regime.

First attempts to utilize series approximations for the unsteady case resulted in failure due to nonconvergence of the series. Faced with a lack of an apparently simple solution to the resulting integrals, numerical integration was resorted to. Here difficulties arose centering around the pathological behavior of the integrands near the origin as pointed out in the Discussion of Results.

As a basic numerical integration technique, the "pyramided" Simpson's rule method was incorporated. For the steady case where the integrals were easily computed, the results of this method were checked against the steady state case already in hand and found to agree perfectly.

The greatest difficulty occurred in the unsteady case for $\lambda = 0$ where one could not reasonably neglect the integrals in the interval near zero. To overcome this problem an analytical approximation (for the integral evaluated near zero) was introduced and checked against a known analytical solution shown in Section III. The technique involved the reduction of the integrals to functions of the Fresnel integrals. However, this procedure resulted in additional problems which were not immediately detected. Although the technique was theoretically sound, the computer numerical error pointed out in the Discussion of Results became apparent after study of the output. Once the cause was detected, the anomaly was remedied in a simple manner.

The coefficients for the unsteady interference functions as presented here shall require a "separate source" check. The Laplace transform solution for the oscillatory transonic wind tunnel interference problem should provide such a check. If these techniques verify one another, a third independent check could be the Hamamoto linearized high frequency solution.

Development of required computer routines for the potential and pressure coefficients from the collocation

and Laplace techniques should be facilitated by the use of subroutines derived for the unsteady coefficient routine presented here.

It is felt that the collocation technique will prove a versatile tool in transonic flow studies in that the freedom which one has in specifying boundary conditions will permit treatment of problems which defy solution by other techniques. For example, the sonic wind tunnel wall interference technique is only good in the π phase shift case while the collocation technique will provide results for any phase relationship. Further, it should be valid for more complex boundary conditions such as those posed by wind tunnels with porous walls.

APPENDIX A

FORMULATION OF THE COLLOCATION COEFFICIENT ROUTINE

A-I. MAIN PROGRAM

The purpose of the main program is the solution of Equations (III-16) and (III-17) for the unknown interference function coefficients.

$$\sum_{N=0}^{L-1}\left\{\mathcal{D}_{ON}X^N+\frac{R}{2}\sqrt{\frac{Q}{\pi}}(\theta_{IN}+\mathcal{D}_{IN})\int_{-1}^{X}\frac{S^N}{(X-S)^{3/2}}\,\mathrm{EXP}\left[\frac{K^2}{Q}(X-S)-\frac{Q}{4}\cdot\frac{R^2}{X-S}\right]dS\right\}=0$$

(III-16)

$$\sum_{N=0}^{L-1}\left\{\mathcal{D}_{IN}X^N+\frac{R}{2}\sqrt{\frac{Q}{\pi}}(\theta_{ON}+\mathcal{D}_{ON})\int_{-1}^{X}\frac{S^N}{(X-S)^{3/2}}\,\mathrm{EXP}\left[\frac{K^2}{Q}(X-S)-\frac{Q}{4}\,\frac{R^2}{X-S}\right]dS\right\}=0$$

(III-17)

Equation (III-16) must be satisfied for each station along the lower blade while simultaneously equation (III-17) must be satisfied at each station along the upper blade. For the purposes of this routine, ten stations have been selected ($X_r = -0.9+0.2r$, $r = 0,1,2,\ldots,9$). Computations near the leading edge are avoided due to the singularity there.

For a given K, λ, and R, the term

$$\frac{R}{2}\sqrt{\frac{Q}{\pi}}\int_{-1}^{X}\frac{S^N}{(X-S)^{3/2}}\mathrm{EXP}\left[\frac{K^2}{Q}(X-S)-\frac{Q}{4}\,\frac{R^2}{X-S}\right]dS = SR(X,N)+iSM(X,N)$$

(A-I-1)

is a complex function of the parameters X and N. The complex value of this function at each (X,N) is provided by Subroutine WSUM and will be dealt with under Appendix A-II. WSUM returns the value of (A-I-1) as real and imaginary parts in arrays SR and SM respectively.

The problem then is to evaluate the various known or computed values and insert them in proper form in a suitable matrix for simultaneous solution.

Observe (III-16) and (III-17) become

$$\sum_{N=0}^{9} \left\{ \mathcal{D}_{ON} X^N + (\theta_{IN} + \mathcal{D}_{IN})(SR(X,N) + iSM(X,N)) \right\} = 0 \tag{A-I-2}$$

and

$$\sum_{N=0}^{9} \left\{ \mathcal{D}_{IN} X^N + (\theta_{ON} + \mathcal{D}_{ON})(SR(X,N) + iSM(X,N)) \right\} = 0 \tag{A-I-3}$$

With θ_{IN} and θ_{ON} determined by boundary conditions at each of the ten stations, (A-I-2) and (A-I-3) form a series of twenty equations in unknowns $\mathcal{D}_{ON}$ and $\mathcal{D}_{IN}$, $N = 0,1,2,\ldots,9$. Note, for the unsteady case, θ_{MN} and $\mathcal{D}_{MN}$ are complex.

Taking the set of coefficients, θ_{MN}, from (III-21), (A-I-2) and (A-I-3) are rewritten for the in phase case as

$$\sum_{N=0}^{9} \left\{ \mathcal{D}_{ON} X^N + \mathcal{D}_{IN}(SR(X,N) + iSM(X,N)) \right\} = (SR(X,0) - DK\ SM(X,1))$$

$$+\ i(SM(X,0) + DK\ SR(X,1)) \tag{A-I-4}$$

and

$$\sum_{N=0}^{9}\left\{\mathcal{D}_{IN}X^N+\mathcal{D}_{ON}(SR(X,N)+iSM(X,N))\right\} = (SR(X,0)-DK\ SM(X,1))$$

$$+\ i(SM(X,0) + DK\ SR(X,1)) \qquad \text{(A-I-5)}$$

where DK replaces K in the computer routine.

For the π phase shift case, the coefficients (III-20) apply and the lower blade boundary equation (A-I-2) becomes

$$\sum_{N=0}^{9}\left\{\mathcal{D}_{ON}X^N+\mathcal{D}_{IN}(SR(X,N)+iSM(X,N))\right\} = -(SR(X,0)-DK\ SM(X,1))$$

$$-\ i(SM(X,0) + DK\ SR(X,1)). \qquad \text{(A-I-6)}$$

The system of twenty equations may be written in matrix form

$$\overline{X}\ \overline{D} = \overline{B} \qquad \text{(A-I-7)}$$

where $\overline{X}$, $\overline{D}$, and $\overline{B}$ are the matrices of coefficients, unknown $\mathcal{D}$'s, and right hand sides. $\overline{X}$, $\overline{D}$, and $\overline{B}$ are complex and may be written

$$\overline{X} = \overline{X}_1 + \overline{X}_2 i$$

$$\overline{D} = \overline{D}_1 + \overline{D}_2 i \qquad \text{(A-I-8)}$$

and

$$\overline{B} = \overline{B}_1 + \overline{B}_2 i.$$

Then in (A-I-7)

$$(\overline{X}_1 + \overline{X}_2 i)(\overline{D}_1 + \overline{D}_2 i) = \overline{B}_1 + \overline{B}_2 i$$

or

$$(\overline{D}_1\overline{X}_1-\overline{D}_2\overline{X}_2) + i(\overline{D}_2\overline{X}_1+\overline{D}_1\overline{X}_2) = \overline{B}_1 + \overline{B}_2 i. \qquad \text{(A-I-9)}$$

It may be concluded that

$$\overline{D}_1\overline{X}_1-\overline{D}_2\overline{X}_2 = \overline{B}_1$$

$$\overline{D}_1\overline{X}_2+\overline{D}_2\overline{X}_1 = \overline{B}_2$$

or

$$\begin{pmatrix} \overline{X}_1 & -\overline{X}_2 \\ \overline{X}_2 & \overline{X}_1 \end{pmatrix} \begin{pmatrix} \overline{D}_1 \\ \overline{D}_2 \end{pmatrix} = \begin{pmatrix} \overline{B}_1 \\ \overline{B}_2 \end{pmatrix} \qquad \text{(A-I-10)}$$

The twenty complex equations in twenty unknowns represented by (A-I-7) have been transformed into 40 real equations in 40 real unknowns. One has only to solve for $\overline{D}_1$ and $\overline{D}_2$ and set

$$\overline{D} = \overline{D}_1 + i\overline{D}_2.$$

PROGRAM STEP ANALYSIS

(Refer to Routine MAIN at the end of this Appendix.)

Double precision is used throughout the routine for

numerical accuracy. The use of the various arrays and common storage areas that are declared will be detailed below.

Various arrays and fixed parameters are initialized in MAIN to minimize computational redundancy in later programming. The arrays COC(10), COS(10), AS(10), and AC(10), zeroized in DO LOOP 100, are utilized in Subroutine WSUM. DL = λ, DK = K, and RHO = R, the controlling parameters, are initialized and PI = π is set. The value for

$$A = \frac{(DK)^2}{(DL)^2 + 4(DK)^2} \qquad \text{(A-I-11)}$$

is computed. For the case DK = 0, A is set to zero to avoid possible computer problems involving division by zero. Additional parameters are computed as follows:

$$B = \frac{DL(RHO)^2}{4} \qquad \text{(A-I-12)}$$

$$C = \frac{DL(DK)^2}{(DL)^2 + 4(DK)^2} \qquad \text{(A-I-13)}$$

$$D = \frac{2(DK)^3}{(DL)^2 + 4(DK)^2} \qquad \text{(A-I-14)}$$

$$E = \frac{DK(RHO)^2}{2} \qquad \text{(A-I-15)}$$

$$G = \frac{1}{2}\,TAN\left(\frac{2DK}{DL}\right) \qquad \text{(A-I-16)}$$

Here G is set to

$$G = \pi/4 \tag{A-I-16a}$$

for the case DL = 0 to avoid division by zero.

$$AK = SIN(G) \tag{A-I-17}$$

$$BK = COS(G) \tag{A-I-18}$$

$$Z = \frac{RHO\ [(DL)^2 + 4(DK)^2]^{1/4}}{2\ \sqrt{PI}} \tag{A-I-19}$$

For future ease in changing the number of stations on the blades, the parameter L is used. Here L = 10 is set. The initial station, $X_1 = -0.9$, and the interval between stations, XINT = 0.2, are set.

The first write statement is included to indicate the controlling parameters DL, DK, and RHO.

Finally, the matrix manipulation section begins. Referencing (A-I-10), array AA(40,40) is dimensioned to hold the coefficient matrix. BB(40) holds the matrix of right hand sides before solution and the matrix of unknowns after solution. (A-I-10) reflects the diagonal duplication of the real and imaginary submatrices, $\overline{X}_1$ and $\overline{X}_2$. Note that equations (A-I-4) and (A-I-5) form adjacent rows in the array and that there is duplication between them for a given X. The first two rows of $\overline{X}_1$ are as follows:

$$1 \quad X_1 \quad \ldots \quad X_1^9 \quad SR(X_1,0) \quad SR(X_1,1) \quad \ldots \quad SR(X_1,9)$$

$$SR(X_1,0) \quad SR(X_1,1) \quad \ldots \quad SR(X_1,9) \quad 1 \quad X_1 \quad \ldots \quad X_1^9$$

(A-I-20)

corresponding to the unknowns

$$\mathcal{D}_{R00} \quad \mathcal{D}_{R01} \quad \ldots \quad \mathcal{D}_{R09} \quad \mathcal{D}_{R10} \quad \mathcal{D}_{R11} \quad \ldots \quad \mathcal{D}_{R19}$$

where $\mathcal{D}_{MN} = \mathcal{D}_{RMN} + i\mathcal{D}_{IMN}$. The first two rows of $\overline{X}_2$ are:

$$0 \quad 0 \quad \ldots \quad 0 \quad SI(X_1,0) \quad SI(X_1,1) \quad \ldots \quad SI(X_1,9)$$

$$SI(X_1,0) \quad SI(X_1,1) \quad \ldots \quad SI(X_1,9) \quad 0 \quad 0 \quad \ldots \quad 0$$

(A-I-21)

corresponding to

$$\mathcal{D}_{I00} \quad \mathcal{D}_{I01} \quad \ldots \quad \mathcal{D}_{I09} \quad \mathcal{D}_{I11} \quad \mathcal{D}_{I12} \quad \ldots \quad \mathcal{D}_{I19}.$$

Reference to (A-I-10) shows the placement of these rows. The first row of $\overline{X}_1$ fills AA(1,1) to AA(1,20) as well as AA(21,21) to AA(21,40). The first row of $\overline{X}_2$ fills AA(1,21) to AA(1,40), and, with signs reversed, AA(21,1) to AA(21,20). For the π phase shift case, the right hand side matrix is filled four rows at a time as follows:

$$BB(1) = -(SR(X_1,0) - DK\ SI(X_1,1))$$

$$BB(2) = (SR(X_1,0) - DK\ SI(X_1,1))$$

$$BB(21) = -(SI(X_1,0) + DK\ SR(X_1,1))$$

$$BB(22) = (SI(X_1,0) + DK\ SR(X_1,1))$$

and (A-I-22)

$$BB(3) = -(SR(X_2,0) - DK\ SI(X_2,1))$$

$$BB(4) = (SR(X_2,0) - DK\ SI(X_2,1))$$

$$BB(23) = -(SI(X_2,0) + DK\ SR(X_2,1))$$

$$BB(24) = (SI(X_2,0) + DK\ SR(X_2,1))$$

and so forth.

DO LOOP 1000 on J = 1,L varies the station from X_1 to X_{10}. It sets two rows in submatrices $\overline{X}_1$ and $\overline{X}_2$, or, equivalently, four rows in AA and four in BB. A is used to set the power of X in submatrix $\overline{X}_1$ and is initially set to $A = X^0 = 1$. JJ is a counter used to select every other row, i.e., JJ = 1,3....,19 as J = 1,2,...,10.

With X_1 set at X = -0.9, WSUM is called to provide the integral term in the boundary equations. WSUM computes the integral for X_1 and for N = 0,1,2,...,9. These ten values are returned in common arrays SR and SM. SR holds the real parts and SM the imaginary as shown in functional notation in (A-I-1). The arrays are filled as follows:

$$
\begin{aligned}
SR(1) &= SR(X,0) & SM(1) &= SM(X,0) \\
SR(2) &= SR(X,1) & SM(2) &= SM(X,1) \\
&\vdots & &\vdots \qquad\qquad (A\text{-}I\text{-}23) \\
SR(10) &= SR(X,9) & SM(10) &= SM(X,9)
\end{aligned}
$$

DO LOOP 900 on I = 1,10 controls the columns. The columns of submatrices $\overline{X}_1$ and $\overline{X}_2$ are filled in sequence for each station counted by DO LOOP 1000. In each step in LOOP 900, four positons in submatrix $\overline{X}_1$ are filled with the power of X. Four positions in $\overline{X}_2$ are zeroized. The real part of the applicable integral is placed in four positions in $\overline{X}_1$ via the dummy variable SUMR. Similarly, the imaginary part is placed into $\overline{X}_2$ via SUMI. Care is taken to obtain the proper signs. Once the LHS is set for each station, four rows in the RHS matrix BB are set according to (A-I-22). The station is then augmented by the interval, XINT, and the procedure is repeated until all positions in AA and BB are filled.

For verification, the matrix prior to solution is printed out. Matrix BB is destroyed in the solution process.

The matrix is solved for the unknowns using library Subroutine DGELG. The inputs are the coefficient matrix, RHS matrix, coefficient matrix size (here set to 40 by M=4L), RHS matrix size, and a significance measure (here set to 1.0E-6). The solution is returned on BB and any error indication on IER.

BB(1) to BB(10) contain the real parts of the coefficients and BB(11) to BB(20) contain the imaginary parts. The parts are combined in complex form and placed in complex array ZB. The form of ZB is

$$
\begin{aligned}
ZB(1) &= DELT(0,0) = \\
ZB(2) &= DELT(0,1) = \\
&\vdots \\
ZB(11) &= DELT(1,0) = \\
&\vdots \\
ZB(20) &= DELT(1,9) =
\end{aligned}
\tag{A-I-24}
$$

Finally the desired coefficients are printed out on computer paper output and on cards.

```
C     ..................................................................
C
C                             MASTER ROUTINE
C
C     ROUTINE FOR THE SOLUTION OF UNSTEADY TRANSONIC CASCADE FLOW.  THE
C     ROUTINE SOLVES FOR THE COEFFICIENTS OF THE INTERFERENCE FUNCTIONS
C     FOR THE UPPER AND LOWER FOIL.  THE METHOD PROVIDES FOR THE
C     SATISFACTION OF THE FLOW TANGENCY REQUIREMENT SIMULTANEOUSLY AT
C     EACH OF TEN POINTS ON THE UPPER AND LOWER FOIL.  INPUTS ARE..
C
C        DL=OSWATITCH'S PARAMETER, LAMBDA
C        DK=STROUHAL NUMBER
C        RHO=SOLIDITY
C
C     TECHNIQUE INVOLVES THE SOLUTION OF 20 COMPLEX EQUATIONS AND 20
C     COMPLEX UNKNOWNS.  COMPLEX MANIPULATION YIELDS A MATRIX (41,40)
C     WHICH IS SOLVED SIMULTANEOUSLY USING LIBRARY SUBROUTINE DGELG.  THE
C     MATRIX PRIOR TO SOLUTION IS PRINTED OUT BY ROW WITH CORRESPONDING
C     RIGHT HAND SIDES.  ERROR INDICATOR IS FROM DGELG AND SHOWS LOSS
C     OF SIGNIFICANCE IF OTHER THAN 0.  THE SOLUTION IS PRINTED IN COMPLEX
C     FORMAT.
C
C        DELT(0,N), DELT(1,N)   N=0,1,...,9, WHERE THE UPPER FOIL
C        INTERFERENCE FUNCTION IS SUM ON N FROM 0 TO 9 OF
C        DELT(1,N)*X**N, AND SIMILARLY FOR THE LOWER FOIL USING DELT(0,N).
C
C     *****180 DEGREE OUT OF PHASE OPERATION***********************
C     ..................................................................
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMPLEX*8 ZB
      DIMENSION AA(40,40),BB(40),ZB(20)
      COMMON AS(10),AC(10),SR(10),SM(10)
      COMMON SS,CC,SSS,CCC,COC(10),COS(10)
      COMMON Z,AK,BK,DK,DL
      COMMON AL,RHO
      COMMON/HOLD/B,C,D,E,EPS,LIM
   51 FORMAT('1','MATRICES PRIOR TO SOLUTION')
  499 FORMAT('0','ROW=',I3/4(' ',10D13.7/)/'+','RHS=',D13.7)
 2000 FORMAT('1','ERROR INDICATION IER=',I3)
 3003 FORMAT('0','DELT(0',I1,')=',2E15.6,20X,'DELT(1',I1,')=',2E15.6)
 3004 FORMAT(4E20.7)
 3010 FORMAT(' ','THE INTERFERENCE COEFFICIENTS')
 3011 FORMAT(4D20.3)
 9000 FORMAT('1','SOLUTIONS FOR THE FOLLOWING PARAMETERS'/'0','DL=',F6.3
     1,10X,'DK=',F6.3,10X,'RHO=',F6.3,10X,'AL=',F6.3//////)
 9001 FORMAT('1'///////' ',20X,'SOLUTIONS FOR THE FOLLOWING PARAMETERS'/
     1'0',20X,'DL=',F6.3,10X,'DK=',F6.3,10X,'RHO=',F6.3)
```

```
 9002 FORMAT('1'////)
C
C
C     INITIALIZATION SECTION
C
C     ..................................................................
C
C     SET THE CONTROLLING VARIABLES   DK,DL,AND RHO
C
      DK=0.0D0
      DL=0.2D0
      RHO=0.5D0
C
C     ..................................................................
C
C     FOR CONVENIENCE ESTABLISH VARIOUS OTHER FIXED VARIABLES THAT ARE
C     FUNCTIONS OF THE CONTROLLING VARIABLES.  IN ADDITION VARIOUS
C     MATRICES ARE ZEROIZED.
C
      AL=0.0D0
      DO 100 I=1,10
      COC(I)=0.0D0
      COS(I)=0.0D0
      AS(I)=0.0D0
  100 AC(I)=0.0D0
      PI=3.141592653589793
      IF(DK.GT.0) GO TO 1
      A=0.0D0
      GO TO 2
    1 A=DK**2/(DL**2+4.0D0*DK**2)
    2 B=DL*RHO**2/4.0D0
      C=DL*A
      D=2.0D0*DK*A
      E=DK*RHO**2/2.0D0
      IF(DL.EQ.0) GO TO 3
      ARG=2.0D0*DK/DL
      G=0.5D0*DATAN(ARG)
      GO TO 4
    3 G=PI/4.0D0
    4 CONTINUE
      AK=DSIN(G)
      BK=DCOS(G)
      Z=RHO*DSQRT(DSQRT(DL**2+4.0D0*DK**2))/(2.0D0*DSQRT(PI))
C     SET THE NUMBER OF STATIONS ON THE FOIL, STATION ONE, AND INTERVAL
      L=10
      X=-0.9D0
      XINT=0.2D0
C     CHECK INPUTS
```

```
      WRITE(6,9002)
      WRITE(6,9001)DL,DK,RHO
C
C     ...............................................................
C     ...............................................................
C
C     BEGIN THE MATRIX MANIPULATION SECTION
C
      P=0.0D0
C
C     SET UP ROW PAIRS KEYING ON X
C
      DO 1000 J=1,L
      A=1.0D0
      JJ=2*J-1
      CALL WSUM(X,J)
C
C     SET UP ROWS KEYING ON COEFFICIENT NUMBER
C
      DO 900 I=1,L
      AA(JJ,I)=A
      AA(JJ+1,I+10)=A
      AA(JJ+20,I+20)=A
      AA(JJ+21,I+30)=A
      AA(JJ,I+20)=P
      AA(JJ+1,I+30)=P
      AA(JJ+20,I)=P
      AA(JJ+21,I+10)=P
      SUMR=SR(I)
      SUMI=SM(I)
      AA(JJ+1,I)=SUMR
      AA(JJ,I+10)=SUMR
      AA(JJ+21,I+20)=SUMR
      AA(JJ+20,I+30)=SUMR
      AA(JJ,I+30)=-SUMI
      AA(JJ+1,I+20)=-SUMI
      AA(JJ+20,I+10)=SUMI
      AA(JJ+21,I)=SJMI
      A=A*X
  900 CONTINUE
C
C     SET VALUES FOR RIGHT HAND SIDE
C
      BB(JJ)=-SR(1)+DK*SM(2)
      BB(JJ+1)=-BB(JJ)
      BB(JJ+20)=-SM(1)-DK*SR(2)
      BB(JJ+21)=-BB(JJ+20)
      X=X+XINT
```

```
 1000 CONTINUE
C
C
C
C     CHECK MATRICES
C
      WRITE(6,51)
      M=4*L
      DO 500 J=1,M
      WRITE(6,499)J,(AA(J,I),I=1,M),BB(J)
  500 CONTINUE
C
C     SOLVE THE SIMULTANEOUS EQUATIONS
C
      CALL DGELG(BB,AA,M,1,1.0E-6,IER)
C     CHECK SOLUTION
      WRITE(6,2000)IER
C
C     FORM COEFFICIENTS
C
      J=2*L
      DO 600 I=1,J
  600 ZB(I)=CMPLX(SNGL(BB(I)),SNGL(BB(I+20)))
C     WRITE ZB
      WRITE(6,9000)DL,DK,RHO,AL
      WRITE(7,3011)DL,DK,RHO,AL
      WRITE(6,3010)
      DO 700 I=1,L
      K=I+10
      N=I-1
      WRITE(6,3003)N,ZB(I),N,ZB(K)
      WRITE(7,3004)ZB(I),ZB(K)
  700 CONTINUE
      STOP
      END
```

A-II. SUBROUTINE WSUM

The purpose of WSUM is to provide the control for the computation of the integral

$$I_e = \frac{R}{2}\sqrt{\frac{Q}{\pi}} \int_{-1}^{X} \frac{S^N}{(X-S)^{3/2}} \mathrm{EXP}\left[\frac{K^2}{Q}(X-S) - \frac{Q}{4}\,\frac{R^2}{X-S}\right] dS. \qquad \text{(A-II-1)}$$

As is demonstrated in Section III-B, (A-II-1) may be reduced to

$$I_e = \frac{R}{2}\sqrt{\frac{Q}{\pi}} \left\{ I_{NR} - iI_{NI} \right\} \qquad \text{(A-II-2)}$$

where

$$I_{NR} = \sum_{J=1}^{N+1} \frac{(-1)^{J-1} N!\, X^{N-J+1}}{(J-1)!\,(N-J+1)!} \int_{0}^{X+1} \Big\{ U^{J-\frac{5}{2}} \mathrm{EXP}[\lambda AU - B/U] \qquad \text{(II-30)}$$

$$\cdot \quad \mathrm{COS}\,[2KAU + E/U] \Big\}\, dU$$

and

$$I_{NI} = \sum_{J=1}^{N+1} \frac{(-1)^{J-1} N!\, X^{N-J+1}}{(J-1)!\,(N-J+1)!} \int_{0}^{X+1} \Big\{ U^{J-\frac{5}{2}} \mathrm{EXP}[\lambda AU - B/U] \qquad \text{(II-31)}$$

$$\cdot \quad \mathrm{SIN}\,[2KAU + E/U] \Big\}\, dU$$

where the fixed parameters A,B,C,D, and E are set in MAIN [Refer to Eqns. (A-I-11, -12, -13, -14, and -15).] and carried to WSUM in common space HOLD.

Given the integral values, the summation is a simple computer manipulation. The solution of these integrals involves three variations of the same basic numerical integration technique. The numerical integration of (III-30) and (III-31) is accomplished by a "pyramided" Simpson's rule using end correction. This technique is presented in Appendix A-III where Subroutine ZSUM is described. ZSUM prepares the numerical solution of the basic integral for whatever values of J,K,λ, and upper and lower limits, XO and XA, are specified.

The difficulty encountered with the numerical integration of this basic integral occurs as the result of the lower limit, XA = 0. For J < 2.5, the amplitude of the integrand may be unbounded near XA = 0. Further, except in the special case where K = 0, the SIN and COS functions in the integrand become unmanageable near zero where the controlling term in their arguments, E/U, becomes large. The resulting infinitesimal period cannot be treated by numerical techniques. For a reasonable λ, one may neglect the troublesome portion of the integral near zero. This is permissable since the exponential function of (-B/U) forces the amplitude to zero in the interval where the integrand is ill-behaved. When λ = 0, for J < 2.5, one may not neglect the integral near zero. Fortunately, in this case, one may approximate the ill-behaved portion of the integral by analytical methods.

For $\lambda = 0$, the basic integrals in Eqns. (III-30) and (III-31) become

$$SUMR = \int_0^{X+1} U^{J-\frac{\zeta}{2}} COS[DU + E/U]dU \qquad (III-33)$$

and

$$SUMI = \int_0^{X+1} U^{J-\frac{\zeta}{2}} SIN[DU + E/U]dU \qquad (III-34)$$

Introducing the correction term of Section III-B, one may write (III-33) and (III-34) as

$$SUMR = COC + \int_{AL}^{X+1} U^{J-\frac{\zeta}{2}} COS[DU + E/U]dU \qquad (A-II-3)$$

and

$$SUMI = COS + \int_{AL}^{X+1} U^{J-\frac{\zeta}{2}} SIN[DU + E/U]dU \qquad (A-II-4)$$

where the correction terms are

$$COC = \int_0^{AL} U^{J-\frac{\zeta}{2}} COS[DU + E/U]dU \qquad (A-II-5)$$

and

$$COS = \int_0^{AL} U^{J-\frac{\zeta}{2}} SIN[DU + E/U]dU \qquad (A-II-6)$$

Integrals COC and COS are prepared by Subroutine COR. The technique as presented in Appendix A-V provides an

approximation which reduces the correction integral to the well known Fresnel integrals.

The limit AL is selected such that (1) the numerical integration technique may be applied to the second term in (A-II-3) and (A-II-4), and (2) the correction approximation is valid over its interval.

STEP ANALYSIS

(Refer to the Subroutine at the end of this Appendix.)

Subroutine WSUM is called to provide the block of L integrals at each station on the blades where the boundary conditions are enforced. (In this investigation L is set at ten.) The subroutine inputs are DK = K, DL = λ, RHO = R, and X.

In the first steps several fixed parameters are established for convenience. EPS = 1.0E-8 is the convergence parameter used in the numerical integration process in Subroutine ZSUM. LIM = 50 is a step limiter also used by ZSUM [Refer to A-III]. PI = π is used in later calculations in WSUM.

A test is made to determine which of the three cases is being dealt with. If K = 0, branching is executed to Step 1. If not, the next step is encountered. If K $\neq$ 0 and λ = 0, the routine branches to step 2. The case K $\neq$ 0 and $\lambda \neq$ 0 follows at step 3.

<u>Step 1, K = 0</u>

The intention here is to use the numerical integration process over the entire interval. Since no SIN or COS

functions enter, one may approach closely to zero at the lower limit (XA is set to 1.0E-8). The upper limit is set to XO = X + 1. Subroutine ZSUM is called to compute the integrals J = 1,2,...,10. The integral in (II-30) is returned on SUMR and the negative of the integral in (II-31) is returned on SUMI. (Note that in this case SUMI = 0.) The values are stored in the vector arrays AC and AS. The routine then jumps to step 1000 where the summation indicated in Eqns. (II-30) and (II-31) takes place.

Step 2, $\lambda = 0$, $K \neq 0$

In this step the special case of $\lambda = 0$ is computed. The Fresnel integral is obtained, and where applicable, the correction is computed and summed with that portion of the integral obtained by numerical methods.

Observing that the value of X does not affect the correction integrals, they need be computed only once. Hence a test is made:

```
IF(I.GT.1) GO TO 22.
```

Step 22 bypasses the correction computation.

The first steps in computing the correction integrals includes the preparation of inputs for Subroutine COR and Subroutine Red. The rationale for these functions is given in A-IV and A-V. They include the following:

AL = RHO/100 (A-II-7)

AR = E/AL (A-II-8)

$$SS = SIN(E/AL) \tag{A-II-9}$$

$$CC = COS(E/AL) \tag{A-II-10}$$

The Fresnel integrals for upper limit AR are obtained from library Subroutine CS. The explanation for CS is given on call from the library and a copy is found at the end of this Appendix. The only argument required is AR. The following is returned.

$$CCC = \int_0^{AR} \frac{COS(T)}{\sqrt{2\pi T}}\, dT \tag{A-II-11a}$$

$$SSS = \int_0^{AR} \frac{SIN(T)}{\sqrt{2\pi T}}\, dT. \tag{A-II-11b}$$

In preparation for their use in Subroutines COR and RED, (A-II-11a,b) are modified as follows:

$$CCC = \sqrt{\frac{2\pi}{E}}\,[\tfrac{1}{2} - CCC] \tag{A-II-12a}$$

and

$$SSS = \sqrt{\frac{2\pi}{E}}\,[\tfrac{1}{2} - SSS]. \tag{A-II-12b}$$

Refer to A-V for an explanation of these functions.

The various inputs in hand, Subroutine COR is called to prepare the block of L = 10 correction integrals for Eqns. (A-II-5, -6). The arguments of COR are AL, D, E, CC, SS,

CCC, and SSS. The value of (A-II-5,-6) for each J is returned on CCOR and SCOR respectively and stored in arrays COC and COS. COC and COS are of vector size ten and are held in common with MAIN for permanency. Also note that (-SCOR) is stored in COR for compatibility with the numerical solution of the remaining integral. The correction terms are printed for reference.

Finally the remainder of the integrals are computed numerically in the same manner as was used in step 1 but with lower limit AL. The result is summed with its correction and stored in AC and AS.

Step 3. $K \neq 0$, $\lambda \neq 0$

In this step, one relies on the exponential function to reduce the amplitude of the integrand near zero. That portion of the integral from zero to 1.0E-3 is neglected. The numerical solution is called as before and the results stored in AC and AS.

Once the values of the integrals for J = 1,2,...,10 are in hand, the indicated summations, Eqns. (III-30,-31), must be carried out. Step 1000 begins this operation. The summation is carried out in a straight forward manner for each of the L = 10 integrals. I_{NR} and I_{NI} are computed on SUMC and SUMS respectively. (Note that IFAC(M) is a function subroutine generating the factorial of M. Refer to A-VI.) The completed integrals are stored in arrays SR and SM and are printed for reference.

Recalling that equation (A-II-2) contained $Q = \lambda+2iK$, a final complex manipulation must be made. Observe that

$$Q = \sqrt{\lambda^2 + 4K^2}\,[\mathrm{COS}(\omega) + i\mathrm{SIN}(\omega)] \qquad \text{(A-II-13)}$$

where

$$\omega = \mathrm{ARCTAN}(2K/\lambda).$$

Then

$$\frac{R}{2}\sqrt{\frac{Q}{\pi}} = Z[BK + iAK] \qquad \text{(A-II-14)}$$

where

$$Z = \frac{R\,[\lambda^2 + 4K^2]^{1/4}}{2\sqrt{\pi}} \qquad \text{(A-I-19)}$$

$$AK = \mathrm{SIN}(G) \qquad \text{(A-I-17)}$$

and

$$BK = \mathrm{COS}(G) \qquad \text{(A-I-18)}$$

for

$$G = \frac{1}{2}\,\mathrm{ARCTAN}(2K/\lambda). \qquad \text{(A-I-16)}$$

Finally, (A-II-2) may be written

$$I_B = Z(I_{NR}BK - I_{NI}AK) + iZ(I_{NR}AK + I_{NI}BK). \qquad \text{(A-II-15)}$$

Z, AK, and BK are computed in MAIN to eliminate repetition and are carried to WSUM in common. DO LOOP 1030 computes the real and imaginary parts of (A-II-15) for the ten integrals and returns these values to MAIN in arrays SR and SM.

```
      SUBROUTINE WSUM(X,I)
C
C     ..................................................................
C
C                    SUBROUTINE WSUM
C
C     SUBROUTINE PROVIDES THE CONTROL FOR THE SOLUTION OF THE INTEGRAL TERM
C     OF THE BOUNDARY EQUATIONS.  INPUTS ARE DK,DL,RHO, AND X=STATION.
C     OUTPUT IS THE EVALUATED INTEGRAL FOR THAT STATION FOR N=0,1,2,...,9.
C     THE COMPLEX VALUES ARE PLACED IN REAL MATRIX SR AND IMAGINARY MATRIX
C     SM IN COMMON WITH THE MASTER PROGRAM.
C
C     THREE CASES ARE DIFFEREINTIATED IN COMPUTING THE BASIC INTEGRALS FOR
C     THE FINAL SUMMATION.  DK=0,DK.NE.0.AND.DL.EQ.0,
C     AND DK.NE.0.AND.DL.NE.0.
C
C     PRINTED OUTPUTS INCLUDE FOR THE CASE DL=0 THE VALUE OF THE FRESNEL
C     INTEGRAL AND THE CORRECTION UPPER LIMIT, THE CORRECTIONS FOR COS
C     AND SIN INTEGRALS, AND THE VALUES OF THE BASIC INTEGRALS INCLUDING
C     THE CORRECTION.
C
C     IN ALL CASES THE VALUE OF THE INTEGRALS AFTER FINAL SUMMATION IS
C     PRINTED OUT.  BEFORE THE VALUES ARE RETURNED TO MAIN A COMPLEX
C     MANIPULATION IS CARRIED OUT.  THE RESULTS OF THIS OPERATION MAY BE
C     OBTAINED FROM THE MATRIX PRIOR TO SOLUTION.
C
C     ..................................................................
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON AS(10),AC(10),SR(10),SM(10)
      COMMON SS,CC,SSS,CCC,COC(10),COS(10)
      COMMON Z,AK,BK,DK,DL
      COMMON AL,RHO
      COMMON/HOLD/B,C,D,E,EPS,LIM
      COMMON/HELD/SI(4),CI(4)
 2100 FORMAT('0'/'0',20X,'DL=0, CORRECTION USED'/'0',20X,'SIN AND COS FR
     1ESNEL INTEGRAL VALUE FOR UPPER LIMIT AR..')
 2110 FORMAT('0','DL=0 COS AND SIN INTEGRAL CORRECTIONS..')
 2119 FORMAT('1')
 2120 FORMAT('0'/'0','BASIC INTEGRALS PRIOR TO SUMMATION, I=',I2)
 2130 FORMAT('0','INTEGRALS AFTER SUMMATION BUT PRIOR TO FINAL COMPLEX M
     1ANIPULATION, I=',I2)
 2300 FORMAT('0',20X,'SSS=',D13.5,10X,'CCC=',D13.5,10X,'AR=',D13.5)
 2500 FORMAT('1','COC='/' ',10D13.5/' ','COS='/' ',10D13.5)
 2501 FORMAT('0','COS INTEGRALS.'/' ',10D13.5)
 2503 FORMAT('0','SIN INTEGRALS.'/' ',10D13.5)
 2505 FORMAT('0','COS..',/' ',10D13.5/'0','SIN..'/' ',10D13.5)
C
```

```
9000 FORMAT(' ',' '/'0',20X,'SOLUTION TO INTEGRAL(X**(J-2.5)SIN OR CO
    1S(E/X) ON LIMITS'/' ',20X,'ZERO TO AL=',1PD10.3,1X,'AND FOR E=',D1
    10.3/'0',20X,' J ',13X,'SIN INTEGRAL',13X,'COS INTEGRAL'//)
C
C
      EPS=1.0D-8
      LIM=50
      PI=3.141592653589793
      IF(DK.EQ.0) GO TO 1
      IF(DL.EQ.0) GO TO 2
      GO TO 3
C     CASE OF K=0  ARGUMENT IS U**(J-5/2)EXP(DLAU-B/U)
C
    1 XA=1.0D-8
      XO=X+1.0D0
      DO 10 J=1,10
      CALL ZSUM(XA,XO,J,SUMR,SUMI)
      AC(J)=SUMR
   10 AS(J)=SUMI
      GO TO 1000
C
C     CASE FOR K.NE.0    DL=0
C
    2 IF(I.GT.1) GO TO 22
C
C     COMPUTE CORRECTION
C
      AL=RHO/100.0D0
      AR=E/AL
      SS=DSIN(AR)
      CC=DCOS(AR)
      CALL CS(CCC,SSS,AR)
      WRITE(6,2100)
      WRITE(6,2300)SSS,CCC,AR
      WRITE(6,9000)AL,E
      ARG=DSQRT(2.0D0*PI/E)
      SSS=ARG*(0.5D0-SSS)
      CCC=ARG*(0.5D0-CCC)
      DO 20 J=1,10
      CALL COR(AL,D,E,J,CC,SS,SSS,CCC,SCOR,CCOR)
      COC(J)=CCOR
      COS(J)=(-SCOR)
   20 CONTINUE
      WRITE(6,2119)
      WRITE(6,2110)
      WRITE(6,2500)(COC(L),L=1,10),(COS(L),L=1,10)
   22 CONTINUE
C
```

```
C      COMPUTE REMAINDER OF THE INTEGRAL
C
       XA=AL
       XO=X+1.0D0
       DO 21 J=1,10
       CALL ZSUM(XA,XO,J,SUMR,SUMI)
       AC(J)=COC(J)+SUMR
       AS(J)=COS(J)+SUMI
   21  CONTINUE
       WRITE(6,2120)I
       WRITE(6,2505)(AC(L),L=1,10),(AS(L),L=1,10)
       GO TO 1000
C
C      CASE FOR K.NE.0, DL.NE.0
C
    3  XA=1.0D-3
       XO=X+1.0D0
       DO 30 J=1,10
       CALL ZSUM(XA,XO,J,SUMR,SUMI)
       AC(J)=SUMR
       AS(J)=SUMI
   30  CONTINUE
       GO TO 1000
C
C      GIVEN BASIC INTEGRALS, CONVERT TO FINAL INTEGRALS
C
 1000  SUMS=0.0D0
       SUMC=0.0D0
       DO 1020 N=1,10
       DO 1010 J=1,N
       ST=(-1.0D0)**(J-1)*IFAC(N-1)*X**(N-J)
       ST=ST/(IFAC(J-1)*IFAC(N-J))
       SUMC=ST*AC(J)+SUMC
 1010  SUMS=ST*AS(J)+SUMS
       SR(N)=SUMC
       SM(N)=SUMS
       SUMC=0.0D0
       SUMS=0.0D0
 1020  CONTINUE
       WRITE(6,2130)I
       WRITE(6,2501)(SR(J),J=1,10)
       WRITE(6,2503)(SM(J),J=1,10)
C
C
C
C
C
C
```

```
C
C     PERFORM FINAL IMAGINARY OPERATION ON INTEGRAL VALUES
C
      DO 1030 J=1,10
      TEMP=SR(J)
      SR(J)=Z*(TEMP*BK-SM(J)*AK)
 1030 SM(J)=Z*(TEMP*AK+SM(J)*BK)
      RETURN
      END
```

```
      SUBROUTINE CS(C,S,X)                                              CS   380
C                                                                       CS    10
C     ...................................................................CS    20
C                                                                       CS    30
C        SUBROUTINE CS                                                  CS    40
C                                                                       CS    50
C        PURPOSE                                                        CS    60
C           COMPUTES THE FRESNEL INTEGRALS.                             CS    70
C                                                                       CS    80
C                                                                       
C     INPUT IS STATION=X, OUTPUTS ARE THE COS INTEGRAL ON C AND SIN
C     INTEGRAL ON S.
C
C     ...................................................................CS   360
C                                                                       CS   370
      Z=ABS(X)                                                          CS   390
      IF(Z-4.)1,1,2                                                     CS   400
    1 C=SQRT(Z)                                                         CS   410
      S=Z*C                                                             CS   420
      Z=(4.-Z)*(4.+Z)                                                   CS   430
      C=C*((((((5.100785E-11*Z+5.244297E-9)*Z+5.451182E-7)*Z            CS   440
     1+3.273308E-5)*Z+1.020418E-3)*Z+1.102544E-2)*Z+1.840965E-1)        CS   450
      S=S*(((((6.677681E-10*Z+5.883158E-8)*Z+5.051141E-6)*Z             CS   460
     1+2.441816E-4)*Z+6.121320E-3)*Z+8.026490E-2)                       CS   470
      RETURN                                                            CS   480
    2 D=COS(Z)                                                          CS   490
      S=SIN(Z)                                                          CS   500
      Z=4./Z                                                            CS   510
      A=(((((((8.768258E-4*Z-4.169289E-3)*Z+7.970943E-3)*Z-6.792801E-3) CS   520
     1*Z-3.095341E-4)*Z+5.972151E-3)*Z-1.606428E-5)*Z-2.493322E-2)*Z    CS   530
     2-4.444091E-9                                                      CS   540
      B=((((((-6.633926E-4*Z+3.401409E-3)*Z-7.271690E-3)*Z+7.428246E-3) CS   550
     1*Z-4.027145E-4)*Z-9.314910E-3)*Z-1.207998E-6)*Z+1.994711E-1       CS   560
      Z=SQRT(Z)                                                         CS   570
      C=0.5+Z*(D*A+S*B)                                                 CS   580
      S=0.5+Z*(S*A-D*B)                                                 CS   590
      RETURN                                                            CS   600
      END                                                               CS   610
```

A-III. SUBROUTINE ZSUM

Subroutine ZSUM computes the value of the integral in the general Eqns. (III-30) and (III-31) by numerical methods.

$$SUMR = \int_{XA}^{XO} U^{J-\frac{5}{2}} EXP[CU-B/U] COS[DU+E/U] dU \quad (A\text{-}III\text{-}1)$$

$$SUMI = \int_{XA}^{XO} U^{J-\frac{5}{2}} EXP[CU-B/U] SIN[DU+E/U] dU \quad (A\text{-}III\text{-}2)$$

The technique utilized is Simpson's rule with end correction.

Noting the behavior of the integrand, especially near XA = 0, it is not satisfactory to compute the integral using a fixed interval. Instead a "pyramided" scheme is used which automatically reduces the size of the subintervals until the scheme can compute the integral to within the limits of a satisfactory convergence criterion.

The general form of Simpson's rule with end correction is

$$SUM = \frac{H}{15}(7Y_1+16Y_2+14Y_3+16Y_4+\ldots+16Y_{N-1}+7Y_N) + \frac{H^2}{15}(Y_1{}'-Y_N{}')$$

or

$$SUM = \frac{H}{15}\left(7(Y_1+Y_N)+16\sum_{i=1}^{\frac{N-1}{2}} Y_{2i}+14\sum_{i=1}^{\frac{N-3}{2}} Y_{2i+1}\right)+\frac{H^2}{15}(Y_1{}'-Y_N{}') \quad (A\text{-}III\text{-}3)$$

where Y_i and Y_i' are the value of the integrand and the derivative of the integrand evaluated at station (i), and H is the length of the subinterval [Refer to Fig. 8].

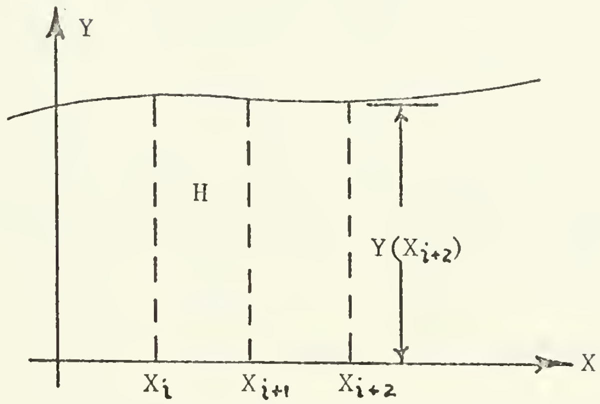


Figure 8. INTEGRATION DIAGRAM

Figure 9 demonstrates the "pyramid" algorithm. X_1 and X_N are the first and last station of the interval over which Y(X) is to be integrated, i.e., X_1 = XA and X_N = XO. N is the number of stations at which the integrand must be evaluated. There are (N-1) equal sub-intervals of length H in INT = $X_N - X_1$.

Step	N	H	X_1								coefficients of X_i								X_N
1	3	INT/2	7								16								7
2	5	INT/4	7				16				14				16				7
3	9	INT/8	7		16		14		16		14		16		14		16		7
4	17	INT/16	7	16	14	16	14	16	14	16	14	16	14	16	14	16	14	16	7

PYRAMID SCHEME
Figure 9

The accuracy of Simpson's approximation increases with decreased H. The pyramided technique progressively computes the integral, doubling the number of sub-intervals each time, until the value of the computation ceases to change to within some desired criterion, EPS.

Figure 9 shows the change in N and H with each step. The coefficients for each step are shown in their relative positions according to station. Observe that once a Y_i is evaluated for a given step, it is used in each succeeding step and need not be reevaluated. Further note that the first time a Y_i is used, its coefficient is 16; thereafter it is 14.

The integrand in (A-III-1) is

$$Y_R(U) = U^{J-\frac{5}{2}}EXP(CU-B/U)COS(DU+E/U), \qquad \text{(A-III-4)}$$

and it may be shown that

$$Y_R{}'(U) = U^{J-\frac{7}{2}}EXP(CU-B/U)\Big\{COS(DU+E/U)(J-5/2+CU+B/U) - (DU-E/U)SIN(DU+E/U)\Big\}. \qquad \text{(A-III-5)}$$

The integrand in (A-III-2) is

$$Y_I(U) = U^{J-\frac{5}{2}}EXP(CU-B/U)SIN(DU+E/U) \qquad \text{(A-III-6)}$$

and

$$Y_I'(U) = U^{J-\frac{7}{2}}EXP(CU-B/U)\Big\{SIN(DU+E/U)(J-5/2+CU+B/U)$$

$$+ (DU-E/U)COS(DU+E/U)\Big\} . \qquad (A\text{-}III\text{-}7)$$

STEP ANALYSIS

(Refer to the subroutine at the end of this section.)

Subroutine ZSUM arguments are as follows:

XA = Lower limit

XO = Upper limit

J = Summation index set in WSUM.[Refer to Eqns. (II-30,-31).]

The following parameters are brought to ZSUM via common space HOLD: B,C,D,E,EPS, and LIM. B,C,D, and E are computed in MAIN [Refer to Eqns. (A-I-12,-13,-14, and -15).]. EPS and LIM are set in WSUM. XA, XO, and J are also set in WSUM. The values of (A-III-1, -2) are returned on SUMR and SUMI.

Equations (A-III-4,-5,-6, and -7) are first evaluated at the left endpoint. AA = CU-B/U is the argument of the exponential function. If AA is less than (-172), it is assumed that the exponential term drives the integrand and its derivative to zero by branching to statement 6 (Computer underflow errors are generated for smaller values of the exponent.). BB = DU+E/U, the argument of the SIN and COS functions, and CC = DU-E/U, a term used in Y_i', are computed. The evaluations of $Y_R(XA)$ and $Y_I(XA)$ are obtained on YAR and YAI from the arithmetic statement functions YR and YI

where YR is (A-III-4) and YI is (A-III-6). EE and FF are additional terms used in the derivative. Finally the values of $Y_R{}'(XA)$ and $Y_I{}'(XA)$ are computed on YARP and YAIP.

Statement 6 sets all values $Y_R(XA)$, $Y_I(XA)$, $Y_R{}'(XA)$, and $Y_I{}'(XA)$ to zero for small values of the exponential function (those cases where AA $\leq$ -172).

The same process is executed for the right end point. Here $Y_R(XO)$ = YOR, $Y_I(XO)$ = YOI, $Y_R{}'(XO)$ = YORP, and $Y_I{}'(XO)$ = YOIP.

The first step in computing the internal points is to set the interval of integration in DINT = XO - XA. Next, various summation parameters are zeroed including SOR and SOI, the sums of the Y_{Ri}'s and Y_{Ii}'s which will be multiplied by the coefficient 14 [Refer to Eqn. (A-III-3)] and SER and SEI, those sums to be multiplied by 16. SP sets the number of subintervals in each step (initially set at 2) and K sets the number of terms in SER and SEI to be computed in each step (initially set at 1).

DO LOOP 100 controls the "pyramid" level or step from one to LIM. H is the length of each subinterval in each step, H = DINT/SP.

On each step, the Y_i's computed the previous step for coefficient 16 (summed in SER and SEI) are added to those with coefficient 14 (SOR and SOI). SER and SEI are then zeroed in preparation for the next computation. XE is the dummy variable in this portion of the calculation. The initial Y_i is computed at XE = XA + H.

DO LOOP 10 computes the K new Y_i's for SER and SEI. AA and BB are computed as before. Y_R(XE) and Y_I(XE) are computed on YER and YEI and summed on SER and SEI respectively. Finally the dummy variable is augmented by 2H and the loop continues.

SUMR and SUMI hold the values of (A-III-1) and (A-III-2) for any given step. TEMPR and TEMPI are used as temporary storage for the values in SUMR and SUMI computed in the previous step. Once a new set of values are computed in SUMR and SUMI, they are compared with the previous values. If both do not differ from the previous values by more than EPS, convergence criteria is met and the loop terminates. Should the process reach LIM iterations without convergence, a warning is printed and the existing values of the integrals are accepted.

```
      SUBROUTINE ZSUM(XA,XO,J,SUMR,SUMI)
C
C     ..............................................................
C
C                       SUBROUTINE ZSUM
C
C     PROGRAM FOR THE COMPUTATION OF THE INTEGRAL
C     U**(J-2.5)EXP(DL*A*U-B/U)SIN OR COS (2KAU+E/U).
C
C     METHOD UTILIZED IS A PYRAMIDED SIMPSON'S RULE WITH END CORRECTION.
C     LIMITS ARE FROM XA TO XO.  PYRAMID LIMIT IS LIM.  CONVERGENCE IS
C     MEASURED BY EPS.  B,C,D,AND E ARE FIXED FUNCTIONS OF DK.DL,AND RHO.
C     TWO INTEGRALS ARE COMPUTED, THE REAL AND IMAGINARY PORTION OF THE
C     INTEGRAL IN THE BOUNDARY EQUATIONS.  THE VALUES ARE RETURNED ON SUMR
C     AND SUMI.
C
C     A WARNING IS GENERATED IN THE CASE OF NON CONVERGENCE.
C
C     ..............................................................
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/HOLD/B,C,D,E,EPS,LIM
      YR(U,J,AA,BB)=U**(J-1)*DEXP(AA)*DCOS(BB)/(U*DSQRT(U))
      YI(U,J,AA,BB)=U**(J-1)*DEXP(AA)*(-DSIN(BB))/(U*DSQRT(U))
    2 FORMAT('1','WARNING-SSUM NON CONVERGENT')
C
C
C     ESTABLISH LEFT END POINT
C
      U=XA
      AA=C*U-B/U
      IF(AA.LE.(-172.0D0)) GO TO 6
      BB=D*U+E/U
      CC=D*U-E/U
      YAR=YR(U,J,AA,BB)
      YAI=YI(U,J,AA,BB)
      EE=DEXP(AA)*U**(J-2)/(U*DSQRT(U))
      FF=DFLOAT(J)-2.5D0+C*U+B/U
      YARP=EE*(DCOS(BB)*FF-CC*DSIN(BB))
      YAIP=(-EE)*(DSIN(BB)*FF+CC*DCOS(BB))
      GO TO 3
    6 YAR=0.0D0
      YAI=0.0D0
      YARP=0.0D0
      YAIP=0.0D0
    3 CONTINUE
C
C
```

```
C
C     ESTABLISH THE RIGHT END POINT
C
      U=XO
      AA=C*U-B/U
      BB=D*U+E/U
      CC=D*U-E/U
      YOR=YR(U,J,AA,BB)
      YOI=YI(U,J,AA,BB)
      EE=DEXP(AA)*U**(J-2)/(U*DSQRT(U))
      FF=DFLOAT(J)-2.5D0+C*U+B/U
      YORP=EE*(DCOS(BB)*FF-CC*DSIN(BB))
      YOIP=(-EE)*(DSIN(BB)*FF+CC*DCOS(BB))
C
C     ESTABLISH INTERVAL
C
      DINT=XO-XA
      SOR=0.0D0
      SOI=0.0D0
      SER=0.0D0
      SEI=0.0D0
      SUMR=0.0D0
      SUMI=0.0D0
C     INITIAL SPACING AND NUMBER OF TERMS IN EVEN SUM
      SP=2.0D0
      K=1
C
C     LOOP  SPECIFYING INTERVAL
C
      DO 100 M=1,LIM
      H=DINT/SP
C     SET OLD EVEN SUM INTO ODD SUM
      SOR=SOR+SER
      SOI=SOI+SEI
      XE=XA+H
C     ZEROIZE EVEN SUM
      SER=0.0D0
      SEI=0.0D0
C
C     LOOP TO ESTABLISH EVEN SUM
C
      DO 10 I=1,K
      AA=C*XE-B/XE
      IF(AA.LE.(-172.0D0)) GO TO 4
      BB=D*XE+E/XE
      YER=YR(XE,J,AA,BB)
      YEI=YI(XE,J,AA,BB)
      GO TO 5
```

```
    4 YER=0.0D0
      YEI=0.0D0
    5 CONTINUE
      SER=SER+YER
      SEI=SEI+YEI
   10 XE=XE+2.0D0*H
C
C     FORM THE INTEGRAL
C
      TEMPR=SUMR
      TEMPI=SUMI
      SUMR=H*(7.0D0*(YAR+YOR)+16.0D0*SER+14.0D0*SOR)/15.0D0
      SUMR=SUMR+H**2*(YARP-YORP)/15.0D0
      SUMI=H*(7.0D0*(YAI+YOI)+16.0D0*SEI+14.0D0*SOI)/15.0D0
      SUMI=SUMI+H**2*(YAIP-YOIP)/15.0D0
      L=M
C
C     TEST FOR CONVERGENCE
C
      TTTR=DABS(TEMPR-SUMR)
      TTTI=DABS(TEMPI-SUMI)
      IF(TTTR.LE.EPS.AND.TTTI.LE.EPS) GO TO 101
C
C     PROVIDE FOR ERROR STATEMENT IF NO CONVERGENCE
C
      IF(M.EQ.LIM) GO TO 1
      GO TO 9
    1 WRITE(6,2)
    9 K=K*2
      SP=SP*2.0D0
  100 CONTINUE
  101 CONTINUE
      RETURN
      END
```

A-IV. SUBROUTINE COR

Subroutine COR controls the computation of the correction integrals (A-II-5) and (A-II-6) for the case $\lambda = 0$. The integrals are

$$CCOR = \int_0^{AL} U^{J-\frac{5}{2}} COS(DU+E/U)dU \qquad \text{(A-IV-1)}$$

and

$$SCOR = \int_0^{AL} U^{J-\frac{5}{2}} SIN(DU+E/U)dU \qquad \text{(A-IV-2)}$$

where D and E are specified in (A-I-14,-15). An analytic approximation for these integrals exists in theory; however, computer numerical error prohibits reliable use of this technique for $J \geq 5$ and $R > 2$ [Refer to the Discussion of Results for an explanation]. For this reason, the approximation is used only for $J \leq 4$. This range includes those troublesome integrals having integrands with infinite amplitude near the origin. Standard numerical integration is used for $J > 4$ where the amplitude of the integrands becomes small near zero thus enabling one to approach zero sufficiently close to ensure that the omitted interval is negligable.

CASE FOR $J \leq 4$

Under the assumption that, for a properly selected AL, as U approaches zero the term E/U dominates DU in the

arguments of SIN and COS then equations (III-40) and (III-41) are reasonable approximations of (A-IV-1) and (A-IV-2) even when a reduced number of terms is included. Such a selection of AL is AL = R/100 which ensures $(E/U)/DU \geq 1.0E4$ for $U \leq AL$.

Subroutine COR carries out the operation of (III-40) and (III-41) where

$$CCOR = \int_0^{AL} U^{J-\frac{5}{2}} COS(E/U)dU - D \int_0^{AL} U^{J-\frac{3}{2}} SIN(E/U)dU$$
$$-\frac{D^2}{2} \int_0^{AL} U^{J-\frac{1}{2}} COS(E/U)dU + \frac{D^3}{6} \int_0^{AL} U^{J+\frac{1}{2}} SIN(E/U)dU \tag{A-IV-3}$$

and

$$SCOR = \int_0^{AL} U^{J-\frac{5}{2}} SIN(E/U)dU + D \int_0^{AL} U^{J-\frac{3}{2}} COS(E/U)dU$$
$$-\frac{D^2}{2} \int_0^{AL} U^{J-\frac{1}{2}} SIN(E/U)dU - \frac{D^3}{6} \int_0^{AL} U^{J+\frac{1}{2}} COS(E/U)dU. \tag{A-IV-4}$$

Due to the inaccuracy in the computation of the integrals by Subroutine RED, a power of U greater than 1.5 is avoided by reducing the number of terms in the truncated series by one each time the series is used. That is, for J = 1, four terms are used; for J = 2, three terms are used; and so forth until only one term is used for J = 4.

CASE FOR J > 4

For this case Subroutine ZSUM is used to numerically compute the integral over the interval XA to AL where XA is picked as close to the origin as computer time and accuracy warrant (XA = 1.0E-4 was seclected for these results). It should be noted, however, that in some cases the value of the integral over this interval may be less than the convergence limit, EPS, used in ZSUM in which case the results may suffer from inaccuracy. Provided the error is on the order of EPS (EPS should be the upper limit) such inaccuracies are not significant and should not affect the end result.

STEP ANALYSIS

(Refer to the subroutine at the end of this Appendix.)

The inputs to Subroutine COR include:

AL = Upper limit of the correction integral

J = Summation index in equation (III-30) and (III-31)

$$D = \frac{2K^3}{\lambda^2 + 4K^2}$$

$$E = \frac{KR^2}{2}$$

C = COS(E/AL)

S = SIN(E/AL)

$$SS = \int_0^{AL} \frac{1}{U^{3/2}} \mathrm{SIN}(E/U)\,dU$$

$$CC = \int_0^{AL} \frac{1}{U^{3/2}} \mathrm{COS}(E/U)\,dU$$

D and E are fixed parameters computed in MAIN; C, S, SS, and CC are formed in Subroutine WSUM. CCOR and SCOR return the values of (A-IV-3) and (A-IV-4) to WSUM.

The first step is to determine if $J = 1$, $2 \leq J \leq 4$, or $J > 4$. For the case $J = 1$, four integrals are computed in (A-IV-3,-4) of the form

$$\mathrm{CINT} = \int_0^{AL} U^{M-\frac{5}{2}} \mathrm{COS}(E/U)\,dU \qquad \text{(A-IV-5)}$$

and

$$\mathrm{SINT} = \int_0^{AL} U^{M-\frac{5}{2}} \mathrm{SIN}(E/U)\,dU \qquad \text{(A-IV-6)}$$

where M = J, J+1, J+2, and J+3. These integrals are easily computed in four step DO LOOP 10 utilizing Subroutine RED. RED reduces (A-IV-5,-6) to the general form

$$\mathrm{Integral} = f(AL,N,E) + \int_0^{AL} \frac{1}{U^{3/2}} {\mathrm{SIN} \atop \mathrm{COS}}(E/U)\,dU. \qquad \text{(A-IV-7)}$$

The values of these integrals are stored in two arrays, SI and CI, of dimension four. Note that should a full four term truncated series be desired for $J > 1$, only one additional integral is needed in (A-IV-3,-4) each time. That is, in terms of the J of the preceding calculation, values of CINT and SINT are required with M = J+1, J+2, J+3, and J+4. Since the first three are already on hand, DO LOOP 13 sets SI(I+1) into SI(I) and CI(I+1) into CI(I). The

fourth position is filled with the newly calculated values of CINT and SINT for M = J+4. However, assuming error in RED for J > 4, it was elected to truncate the series by one additional term for each case, J = 2,3, and 4. Hence, zero is set into SI(4) and CI(4).

One assumes that this reduction in accuracy is still acceptable compared to the error introduced by an attempt to use numerical integration.

In the case J > 4, a branch is provided to statement 21 where subroutine ZSUM is used to calculate the integral between AL and XA. Although this procedure simplifies the handling of COR results by WSUM, inaccuracies of the order of EPS are expected as previously mentioned. One might expect greater accuracy using another technique which did not compute the integral piecemeal.

```
      SUBROUTINE COR(AL,D,E,J,C,S,SS,CC,SCOR,CCOR)
C
C     ..................................................................
C
C                          SUBROUTINE COR
C     COMPUTATION OF THE CORRECTION TO INTEGRAL U**(J-2.5)*COS OR
C     SIN(DU+E/U) FOR THE CASE OF DL=0 AND J=1 THROUGH 5.  NUMERICAL
C     INTEGRATION IS USED FOR CASES J=6 THROUGH TEN
C
C     THE CORRECTION IS FOR THE ILLBEHAVED PORTION OF THE INTEGRAL NEAR
C     ZERO WHICH CANNOT BE COMPUTED BY NUMERICAL METHODS.  LIMITS ON THE
C     CORRECTION ARE ZERO TO AL.  D AND E ARE FIXED FUNCTIONS OF DK AND
C     RHO.  J IS A FUNCTION OF N.  S AND C ARE THE FUNCTIONS SIN(E/AL)
C     AND COS(E/AL).  SS AND CC ARE FUNCTIONS OF THE BASIC FRESNEL
C     INTEGRALS.
C
C     THE METHOD UTILIZES A TRUNCATED SIN AND COS SERIES APPROXIMATION
C     AND A REDUCTION SUBROUTINE WHICH REDUCES THE INTEGRAL TO A BASIC
C     FUNCTION OF THE FRESNEL INTEGRAL.
C
C     THE VALUES OF THE SIN AND COS INTEGRAL CORRECTION ARE RETURNED ON
C     SCOR AND CCOR RESPECTIVELY
C
C     PRINTED OUTPUT INCLUDES THE UPPER LIMIT OF THE CORRECTION, THE
C     N FUNCTION J, AND THE VALUES OF THE BASIC CORRECTION INTEGRALS
C     BEFORE TREATMENT IN THE TRUNCATED SERIES.  THE FINAL CORRECTION
C     AFTER THE SERIES OPERATION IS PRINTED OUT BY SUBROUTINE WSUM.
C
C     ..................................................................
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/HELD/SI(4),CI(4)
C
C     TEST FOR CASES J GREATER THAN 6, GREATER THAN TWO OR EQUAL TO ONE
C
      IF(J.GE.5) GO TO 21
      IF(J.GE.2) GO TO 12
C
C     CASE J EQUAL ONE
C     UTILIZE THE REDUCTION FORMULA TO OBTAIN THE BASIC CORRECTION TERMS.
C
      K=J
      DO 10 I=1,4
      CALL RED(AL,E,K,S,C,SS,CC,SINT,CINT)
      SI(I)=SINT
      CI(I)=CINT
```

```
   10 K=K+1
      GO TO 14
   12 DO 13 I=1,3
      SI(I)=SI(I+1)
   13 CI(I)=CI(I+1)
C
C
C
C     CASES J BETWEEN THREE AND FIVE
C     SET INTEGRALS TO ZERO FOR THOSE CASES WHERE COMPUTER
C     NUMERICAL ERROR IS EXPECTED
C
      SI(4)=0.0D0
      CI(4)=0.0D0
   14 CONTINUE
C
C     UTILIZE THE SIN AND COS TRUNCATED SERIES TO OBTAIN THE FINAL
C     INTEGRAL CORRECTION.
C
      SCOR=SI(1)+D*(CI(2)-D*(SI(3)/2.0D0+D*CI(4)/6.0D0))
      CCOR=CI(1)-D*(SI(2)+D*(CI(3)/2.0D0-D*SI(4)/6.0D0))
      GO TO 30
C
C     PROVIDE INTEGRAL CORRECTION BY NUMERICAL INTEGRATION TECHNIQUES
C     FOR THOSE REGIONS WHERE ANALYTICAL SOLUTION IS NOT POSSIBLE
C     DUE COMPUTER NUMERICAL ERROR
C
   21 XA=1.0D-4
      XO=AL
      CALL ZSUM(XA,XO,J,CCOR,SCOR)
      SCOR=-SCOR
   30 CONTINUE
      RETURN
      END
```

A-V. SUBROUTINE RED

Subroutine RED provides the value of the general integrals

$$SINT = \int_0^A U^{J-\frac{5}{2}} SIN(E/U)dU \qquad (A-V-1)$$

and

$$CINT = \int_0^A U^{J-\frac{5}{2}} COS(E/U)dU \qquad (A-V-2)$$

by reducing them to functions of the form

$$SINT = F_1(A,J,E) + \int_0^A \frac{1}{U^{3/2}} \begin{matrix} SIN \\ COS \end{matrix} (E/U)dU \qquad (A-V-3)$$

and

$$CINT = F_2(A,J,E) + \int_0^A \frac{1}{U^{3/2}} \begin{matrix} SIN \\ COS \end{matrix} (E/U)dU. \qquad (A-V-4)$$

The integrals in (A-V-3,-4) are functions of the Fresnel integrals as follows:

$$CC = \int_0^A \frac{1}{U^{3/2}} COS(E/U)dU = \sqrt{\frac{2\pi}{E}}[\frac{1}{2} - C(E/A)] \qquad (A-V-5)$$

and

$$SS = \int_0^A \frac{1}{U^{3/2}} SIN(E/U)dU = \sqrt{\frac{2\pi}{E}}[\frac{1}{2} - S(E/A)] \qquad (A-V-6)$$

where

$$C(E/A) = \int_0^{E/A} \frac{1}{\sqrt{2\pi T}} \cos(T)\,dT \qquad \text{(A-V-7)}$$

and

$$S(E/A) = \int_0^{E/A} \frac{1}{\sqrt{2\pi T}} \sin(T)\,dT. \qquad \text{(A-V-8)}$$

That is, SINT and CINT for any J may be reduced by integration by parts to functions of the basic Fresnel integrals.

Integration by parts provides the following relations for (A-V-1).

If J = 1,

$$\text{CINT} = \int_0^{A} \frac{1}{U^{3/2}} \cos(E/U)\,dU. \qquad \text{(A-V-9)}$$

For J = 2,

$$\text{CINT} = 2[A^{1/2}\cos(E/A) - E\int_0^{A} \frac{1}{U^{3/2}}\sin(E/U)\,dU]. \qquad \text{(A-V-10)}$$

And for $J \geq 3$,

$$\text{CINT} = \frac{(J-2)!}{(2J-3)!}\left\{A^{1/2}\cdot\right.$$

$$\cdot\left[\cos(E/A)\sum_{N=1}^{[\frac{J}{2}+.1]}\frac{[2(J-2N)]!(-1)^{N-1}2^{4N-3}A^{J-2N}E^{2(N-1)}}{(J-2N)!}\right.$$

$$\left.-\sin(E/A)\sum_{N=1}^{[\frac{J}{2}-.1]}\frac{[2(J-2N-1)]!(-1)^{N-1}2^{4N-1}A^{J-2N-1}E^{2N-1}}{(J-2N-1)!}\right]$$

$$\left.+\ \text{INT}\right\} \qquad \text{(A-V-11)}$$

where

$$\text{INT} = 2^{2J-3}E^{J-1}(-1)^{[\frac{J}{2}]}\int_0^A \frac{1}{U^{3/2}}\cos(E/U)\,dU,\quad (-1)^J < 0 \qquad \text{(A-V-12a)}$$

or

$$\text{INT} = 2^{2J-3}E^{J-1}(-1)^{(\frac{J}{2})}\int_0^A \frac{1}{U^{3/2}}\sin(E/U)\,dU,\quad (-1)^J > 0. \qquad \text{(A-V-12b)}$$

Similarly, it may be shown that for (A-V-2) one obtains:

For J = 1,

$$\text{SINT} = \int_0^A \frac{1}{U^{3/2}}\sin(E/U)\,dU, \qquad \text{(A-V-13)}$$

if J = 2,

$$\text{SINT} = 2[A^{\frac{1}{2}}\sin(E/A) + E\int_0^A \frac{1}{U^{3/2}}\cos(E/U)\,dU], \qquad \text{(A-V-14)}$$

and for $J \geq 3$,

$$\text{SINT} = \frac{(J-2)!}{(2J-3)!}\Bigg\{A^{1/2}\cdot$$

$$\cdot\Bigg[\text{SIN}(E/A)\sum_{N=1}^{[\frac{J}{2}+.1]}\frac{[2(J-2N)]!(-1)^{N-1}2^{4N-3}A^{J-2N}E^{2(N-1)}}{(J-2N)!}$$

$$+\ \text{COS}(E/A)\sum_{N=1}^{[\frac{J}{2}-.1]}\frac{[2(J-2N-1)]!(-1)^{N-1}2^{4N-1}A^{J-2N-1}E^{2N-1}}{(J-2N-1)!}\Bigg]$$

$$+\ \text{INT}\Bigg\} \qquad \text{(A-V-15)}$$

where

$$\text{INT} = 2^{2J-3}E^{J-1}(-1)^{[\frac{J}{2}]}\int_0^A \frac{1}{U^{3/2}}\text{SIN}(E/U)dU, \quad (-1)^J < 0 \qquad \text{(A-V-16a)}$$

or

$$\text{INT} = 2^{2J-3}E^{J-1}(-1)^{(\frac{J}{2}-1)}\int_0^A \frac{1}{U^{3/2}}\text{COS}(E/U)dU, \quad (-1)^J > 0. \qquad \text{(A-V-16b)}$$

The notation [ARG] indicates the sign of ARG with the largest integer less than the absolute value of ARG. The use of [ARG ± 0.1] ensures the proper value is selected and excludes the possibility of computer round off error.

STEP ANALYSIS

(Refer to the subroutine at the end of this Appendix.)

The arguments of Subroutine RED include:

A = Upper limit of the correction integral.

$$E = KR^2/2$$

J = Related to the summation index in (III-30,-31)

$$S = SIN(E/A)$$

$$C = COS(E/A)$$

$$SS = \int_0^A \frac{1}{U^{3/2}} SIN(E/U)dU$$

$$CC = \int_0^A \frac{1}{U^{3/2}} COS(E/U)dU$$

E is set in MAIN and carried in common to WSUM and as an argument through COR to RED. A,S,C,SS, and CC are formed in Subroutine WSUM and are carried as arguments to RED via COR. J is set by COR. The values of (A-V-1,-2) are returned to Subroutine COR on SINT and CINT.

A test is made to determine if $J < 3$. If it is, the routine branches to statement 1000.

If $J \geq 3$, equations (A-V-11) and (A-V-15) apply. I1 and I2 are the summation limits where INT(ARG) = [ARG] described previously. The primary series DO LOOP computes the first summation in (A-V-11, -15). The secondary series DO LOOP computes the second summation. A test is made to check the criteria of (A-V-12,-16) and proper branching is initiated. Statement 31 computes (A-V-12a,-16a) for $(-1)^J < 0$, and statement 32 computes (A-V-12b,-16b) for $(-1)^J > 0$. Finally, a statement 40 combines the terms to form (A-V-11,-15) on SINT and CINT.

For the case $J < 3$, a test is made to further restrict the case to $J = 2$ or $J = 1$. For $J = 2$, (A-V-10,-14) are used. For $J = 1$, (A-V-9,-13) are computed.

```
      SUBROUTINE RED(A,E,J,S,C,SS,CC,SINT,CINT)
C
C     ..............................................................
C
C                        SUBROUTINE RED
C
C     COMPUTE INTEGRAL X**(J-2.5)SIN OR COS(E/X) BY REDUCTION TO
C     FRESNEL INTEGRAL
C
C     METHOD UTILIZES A REDUCTION FORMULA DEVELOPED FROM INTEGRATION
C     BY PARTS.  LIMITS ARE ZERO TO A.  E IS A FIXED FUNCTION ON DK AND
C     RHO.  S AND C ARE SIN AND COS FUNCTIONS OF E AND A.  SS AND CC ARE
C     FUNCTIONS OF THE BASIC FRESNEL INTEGRALS.  THE VALUES OF THE
C     COMPUTATION ARE RETURNED ON SINT AND CINT FOR THE SIN AND COS
C     INTEGRAL RESPECTIVELY.
C
C     ..............................................................
C
      IMPLICIT REAL*8(A-H,O-Z)
C
C
      IF(J.LT.3) GO TO 1000
C
C     CASE OF J.GT.2
C
      I1=INT(FLOAT(J)/2.0+0.1)
      I2=INT(FLOAT(J)/2.0-0.1)
      SUMP=0.0D0
      SUMS=0.0D0
C
C     PRIMARY SERIES DO LOOP
C
      DO 10 N=1,I1
      KK=J-2*N
      TERM=DFLOAT(IFAC(2*KK))/DFLOAT(IFAC(KK))
      TERM=TERM*(-1.0D0)**(N-1)*2.0D0**(4*N-3)
      TERM=TERM*A**KK*E**(2*(N-1))
   10 SUMP=SUMP+TERM
C
C     SECONDARY SERIES DO LOOP
C
      DO 20 N=1,I2
      KK=J-2*N-1
      TERM=DFLOAT(IFAC(2*KK))/DFLOAT(IFAC(KK))
      TERM=TERM*(-1.0D0)**(N-1)*2.0D0**(4*N-1)
```

```
      TERM=TERM*A**KK*E**(2*N-1)
   20 SUMS=SUMS+TERM
C
C     FORM INTEGRAL PART OF SUM
C     TEST FOR ODD OR EVEN J
C
      KK=(-1)**J
      IF(KK)31,32,32
C
C     ODD J
C
   31 KK=INT(FLOAT(J)/2.0)
      SUB=2.0D0**(2*J-3)*E**(J-1)*(-1.0D0)**KK
      SINT=SUB*SS
      CINT=SUB*CC
      GO TO 40
C
C     EVEN J
C
   32 SUB=2.0D0**(2*J-3)*E**(J-1)
      SINT=SUB*(-1.0D0)**(J/2-1)*CC
      CINT=SUB*(-1.0D0)**(J/2)*SS
C
C     FORM THE CORRECTION SUMMATION
C
   40 SUB=DFLOAT(IFAC(J-2))/DFLOAT(IFAC(2*J-3))
      B=DSQRT(A)
      SINT=SUB*(B*(S*SUMP+C*SUMS)+SINT)
      CINT=SUB*(B*(C*SUMP-S*SUMS)+CINT)
      GO TO 3000
 1000 IF(J.LT.2) GO TO 2000
C
C     CASE OF J=2
C
      Z=DSQRT(A)
      SINT=2.0D0*(Z*S+E*CC)
      CINT=2.0D0*(Z*C-E*SS)
      GO TO 3000
C
C     CASE OF J=1
C
 2000 SINT=SS
      CINT=CC
 3000 CONTINUE
      WRITE(6,9000)J,SINT,CINT
 9000 FORMAT('0',20X,I3,10X,1PD15.4,10X,D15.4)
      RETURN
      END
```

A-VI. SUBROUTINE IFAC

Function subroutine IFAC returns the value of the argument factorialed, e.g., IFAC(5) = $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. It has provisions for IFAC(0) = 1. A copy of the subroutine follows.

```
      FUNCTION IFAC(I)
C
C     ..........................................................................
C                                SUBROUTINE IFAC
C
C     IFAC IS A FUNCTION SUBROUTINE DESIGNED TO PRODUCE FACTORIALS
C
C     ..........................................................................
C
      IF(I.EQ.0) GO TO 11
      IF(I.EQ.1) GO TO 11
      IFAC=I
      II=I-1
      DO 10 J=1,II
   10 IFAC=IFAC*(I-J)
      GO TO 12
   11 IFAC=1
   12 RETURN
      END
```

APPENDIX B

STEADY STATE POTENTIAL SOLUTION

B-I. COLLOCATION TECHNIQUE

In order to investigate the perturbation velocity potential along a cascade blade for the steady state and π phase shift in adjacent blades, one may use the following:

$$\psi_{K=0}(X,0) = \frac{1}{\sqrt{\pi\lambda}}\left\{\int_{-1}^{X} \frac{1}{\sqrt{X-S}}\, dS + \int_{-1}^{X} \frac{1}{\sqrt{X-S}} \mathrm{EXP}\left[-\frac{\lambda\, R^2}{4(X-S)}\right] dS\right\}$$

$$+ \frac{1}{\sqrt{\pi\lambda}} \sum_{N=0}^{L-1}\left\{\mathcal{D}_{IN} \int_{-1}^{X} \frac{S^N}{\sqrt{X-S}}\, \mathrm{EXP}\left[-\frac{\lambda\, R^2}{4(X-S)}\right] dS\right.$$

$$\left. - \mathcal{D}_{ON} \int_{-1}^{X} \frac{S^N}{\sqrt{X-S}}\, dS\right\} \qquad \text{(III-47)}$$

The second and third integrals are of the form

$$I = \int_{-1}^{X} \frac{S^N}{\sqrt{X-S}} \mathrm{EXP}[-B/(X-S)]\, dS, \quad \text{where } B = \lambda R^2/4. \qquad \text{(B-I-1)}$$

Letting

$$U = B/(X-S), \qquad \text{(B-I-2)}$$

one may write (B-I-1) as

$$I = \sqrt{B} \int_{\frac{B}{X+1}}^{\infty} \frac{1}{U^{3/2}} (X-B/U)^N\, e^{-U}\, dU \qquad \text{(B-I-3)}$$

or

$$I = \sqrt{B}\, X^N \int_{\frac{B}{X+1}}^{\infty} \frac{1}{U^{3/2}} (1-B/XU)^N\, e^{-U}\, dU. \qquad \text{(B-I-4)}$$

The binomial expansion provides

$$(1-B/XU)^N = \sum_{K=0}^{N} \frac{(-1)^K\, N!}{(N-K)!\,K!} (B/XU)^K. \qquad \text{(B-I-5)}$$

Hence (B-I-4) becomes

$$I = \sqrt{B}\, X^N N! \sum_{K=0}^{N} \frac{(-1)^K B^K}{(N-K)!\,K!\,X^K} \int_{\frac{B}{X+1}}^{\infty} U^{-(K+\frac{3}{2})}\, e^{-U}\, dU. \qquad \text{(B-I-6)}$$

It may be shown that integration by parts provides

$$\int_{F^2}^{\infty} \frac{1}{X^m} e^{-X}\, dX = \left[-\frac{1}{M-1} \frac{e^{-X}}{X^{m-1}}\right]_{F^2}^{\infty}$$

$$- \frac{1}{M-1} \int_{F^2}^{\infty} \frac{e^{-X}}{X^{m-1}}\, dX, \quad M > 0 \qquad \text{(B-I-7)}$$

where M may be any half odd integer greater than zero, e.g., M = 1/2, 3/2, 5/2, etc. Continuation of the integration process yields a general relationship

$$\int_{F^2}^{\infty} \frac{e^{-X}}{X^m} dX = \left[e^{-F^2} \sum_{I=1}^{m-\frac{1}{2}} \frac{(M-I-1)!\,(-1)^{I-1}}{(M-1)!\,(F^2)^{m-I}}\right]$$

$$+ \frac{(-1)^{m-\frac{1}{2}} \sqrt{\pi}\, \mathrm{ERFC}(F)}{(M-1)!} \qquad \text{(B-I-8)}$$

where

$$\mathrm{ERFC}(F) = \frac{1}{\sqrt{\pi}} \int_{F^2}^{\infty} \frac{e^{-U}}{U^{1/2}}\, dU = \frac{1}{\sqrt{\pi}}[1 - \mathrm{erf}(F)]. \qquad \text{(B-I-9)}$$

For the case N = 0, (B-I-6) is

$$I = \sqrt{B} \int_{\frac{B}{X+1}}^{\infty} \frac{e^{-U}}{U^{3/2}}\, dU. \qquad \text{(B-I-10)}$$

The last integral in (III-47) may be written

$$\int_{-1}^{X} \frac{S^N}{\sqrt{X-S}}\, dS = 2\sqrt{X+1} \sum_{K=0}^{N} \frac{X^{N-K}(-1)^K N!(X+1)^K}{(2K+1)K!(N-K)!}, \qquad \text{(B-I-11)}$$

and observe that for the case N = 0, (B-I-11) becomes

$$\int_{-1}^{X} \frac{1}{\sqrt{X-S}}\, dS = 2\sqrt{X+1}. \qquad \text{(B-I-12)}$$

Combining Eqns. (B-I-6,-10,-11, and -12) in (III-47) and replacing N with J-1 and K with K-1, one arrives at the final computer compatible equation

$$\psi_{K=0}(X,0) = \frac{1}{\sqrt{\pi\lambda}}[2\sqrt{X+1} + 2\sqrt{B}\ \mathrm{SSUM}[-3/2,\ b/(X+1)]$$

$$- \sum_{J=1}^{L} \sum_{K=1}^{J} \frac{2(-1)^{K-1}(J-1)!X^{J-K}}{(K-1)!(J-K)!} \Big\{ \mathcal{D}_{0,J-1} \frac{(X+1)^{K-1}\sqrt{X+1}}{2K-1}$$

$$- \mathcal{D}_{1,J-1} B^{K-1}\sqrt{B}\ \mathrm{SSUM}[-(K+\tfrac{1}{2}),B/(X+1)]\Big\}] \qquad \text{(B-I-13)}$$

where

$$SSUM(A,FSQ) = \frac{1}{2} \int_{FSQ}^{\infty} X^{A} EXP(-X)dX. \qquad \text{(B-I-14)}$$

SSUM is computed in Subroutine SSUM and will be dealt with in Appendix B-I-B.

B-I-A. THE MAIN PROGRAM

The purpose of MAIN is the computation of (B-I-13). Inputs to the program are the controlling parameters, blade spacing, R, and thickness parameter, λ, as well as the values of the interference coefficients obtained from the collocation coefficient solution of Appendix A.

Step Analysis

(Refer to the routine at the end of Appendix B-I.)

Double precision is used in Subroutine SSUM and single precision is used in MAIN. Double precision arguments to and from SSUM are carried on variables beginning with W.

The inputs DLAM = λ, DK = K, and RHO = R, and the coefficients of the upper and lower interference functions ($\mathcal{D}_{1m}$ and $\mathcal{D}_{0m}$, m = 1,2,...,10) are read into the routine from data cards and printed out immediately for verification. (Note that K, the reduced frequency, is used for accounting purposes on the output only.) Array DEL(20) carries the coefficients with the lower blade coefficients in the first ten positions.

The inputs positioned, various fixed parameters are formed and the station, X, is initialized.

$$B = \lambda R^2/4$$

$$BH = \sqrt{B}$$

$$C = 1/\sqrt{\pi\lambda}$$

$$D = -3/2$$

$$X = -0.9$$

DO LOOP 100 is the station control loop which generates a potential value at each of 20 positions on the blade ($X = -0.9, -0.8, \ldots, 1.0$).

In preparation for the use of Subroutine SSUM, the lower limit is formed,

$$WFSQ = B/(X+1).$$

SSUM is called to form

$$SSUM[D,B/(X+1)] = \frac{1}{2} \int_{\frac{B}{X+1}}^{\infty} \frac{e^{-U}}{U^{3/2}}\, dU.$$

With $AA = \sqrt{X+1}$, the first two terms in (B-I-13) are formed:

$$BB = 2\sqrt{X+1} + 2\sqrt{B}\, SSUM[-3/2,B/(X+1)].$$

DO LOOP 200 accomplishes the J summation in (B-I-13) while LOOP 300 carries out the K summation. (Recall that

L = 10.) Each term in the summation is built up in a straightforward manner. Subroutine SSUM is required to compute

$$\mathrm{SSUM}(A,\mathrm{WFSQ}) = \frac{1}{2} \int_{\mathrm{WFSQ}}^{\infty} U^{A} e^{-U}\, dU,$$

where

$$A = -(K+\tfrac{1}{2}).$$

Function subroutine IFAC is also used. This subroutine produces the factorials of its argument and is explained in Appendix A-VI.

The summation loop completed, the summation term is combined with BB and C to form POT = $\psi_{K=0}(X,0)$ of equation (B-I-13). Finally POT is printed together with its station.

B-I-B. SUBROUTINE SSUM

The purpose of SSUM is to produce the value of the general integral

$$\frac{1}{2} \int_{F^2}^{\infty} X^{A}\mathrm{EXP}(-X)dX, \quad A = \pm 1/2,\ \pm 3/2,\ \pm 5/2, \ldots \qquad \text{(B-I-15)}$$

Only the derivation of the procedure for A < 0 will be presented in detail here. The case for A > 0 is derived in a similar manner. This latter case is not of interest in the potential solution.

The on line function subroutine, INT(A), generates the sign of A with the largest integer less than A, e.g., INT(-3/2) = -1. The subroutine, ERFC(F), produces the compliment of the error function using F^2 as the lower limit. It should be understood that $(N/2)! = \frac{N}{2} \cdot (\frac{N}{2} - 1) \cdot (\frac{N}{2} - 2) \cdots \frac{1}{2}$ and that $(-1/2)! = 1$.

Now for demonstration purposes, (B-I-8) is expanded for the cases A = -3/2 and A = -5/2.

<u>A = - 3/2</u>

$$\frac{1}{2} \int_{F^2}^{\infty} \frac{e^{-X}}{X^{3/2}}\, dX = \frac{1}{2}\left[\frac{1}{e^{F^2}} \frac{2}{F} - 2\sqrt{\pi}\,\mathrm{ERFC}(F)\right] \qquad \text{(B-I-16)}$$

<u>A = - 5/2</u>

$$\frac{1}{2} \int_{F^2}^{\infty} \frac{e^{-X}}{X^{5/2}}\, dX = \frac{1}{2}\left[\frac{1}{e^{F^2}}\left\{\frac{2}{3F^3} - \frac{2^2}{3.1F}\right\} + \frac{2^2\sqrt{\pi}\,\mathrm{ERFC}(F)}{3.1}\right] \qquad \text{(B-I-17)}$$

A routine may be generated in the following manner. Observe that if one defines I = INT(A), there are I terms in the series (Those terms not including the error function term.). The numerator is increased by two in each succeeding term. The denominator of the first term includes (2I-1); each succeeding denominator in the series is multiplied in turn by (2I-1)-2, (2I-1)-4, ... The power of F in the denominator begins with (2I-1) and is decreased by two in each succeeding term. The sign changes with each term. Also

note that the numerical coefficient of the error function term is the same as that for the last series term with the sign reversed.

Step Analysis

(Refer to the subroutine at the end of Appendix B-I.)

The inputs to the subroutine are A and FSQ = F^2. The value of (B-I-15) is returned on SUM. In the first steps various fixed parameters are formed, the summation parameter is zeroed, and INT(A) is determined. A test is made to determine if A > 0 or A $\leq$ 0. Here only the branch A $\leq$ 0 is of interest although the structure of the unused portion of the subroutine for A > 0 is similar.

For A $\leq$ 0, statement 10 tests for the special case $A = \frac{1}{2}$ (or I = 0). This case cannot be generated by the algorithm used for other cases. If $A = -\frac{1}{2}$, statements 12 and 14 set

$$\text{SUM} = \frac{\sqrt{\pi}}{2}\,\text{ERFC(F)}$$

and the routine returns to MAIN.

If $A < -\frac{1}{2}$, statement 13 begins the algorithm. After several preliminary steps, DO LOOP 11 forms the series pointed out previously. The method takes advantage of the observations made in the introductory material. The series formed on SUM, it is divided by the exponential and the error function term is added.

The result is returned to MAIN.

```
C
C     ***************************************************************
C
C     PROGRAM FOR THE COMPUTATION OF THE STEADY STATE POTENTIAL
C     ALONG THE BLADE SURFACE.  INPUTS ARE THE REAL COEFFICIENTS
C     OF THE INTERFERENCE FUNCTION AND THE SET OF PARAMETERS
C     DL,DK,AND RHO.  OUTPUT IS THE NUMERICAL VALUE OF THE POTENTIAL
C     AT EACH STATION ALONG THE UPPER SURFACE OF THE BLADE.
C
C     ***************************************************************
C
      IMPLICIT REAL*8 (W)
      REAL*8 DBLE
      DIMENSION DEL(20)
    5 FORMAT('0',20X,'THE INTERFERENCE COEFFICIENTS'//)
   10 FORMAT(E20.7,20X.E20.7)
   15 FORMAT(' ',20X,'DEL(0',I1,')=',1PE14.7,10X,'DEL(1',I1,')=',E14.7)
   30 FORMAT(3D20.3)
   31 FORMAT('0'/'0',20X,'VALUE OF THE POTENTIAL ALONG THE BLADE'/'0',20
     1X,'STATION',13X,'POTENTIAL')
   32 FORMAT('1'//////' ',20X,'POTENTIAL SOLUTION FOR THE FOLLOWING PARA
     1METERS'/'0',20X,'DL=',F7.3,10X,'DK=',F7.3,10X,'RHO=',F7.3/)
  400 FORMAT('0',20X,'X=',F5.2,10X,'POT=',1PE14.7)
 1000 FORMAT('1')
C
C     INPUT THE VARIABLES AND COEFFICIENTS
C
      READ(5,30)DLAM,DK,RHO
      DO 20 I=1,10
      K=I+10
      READ(5,10)DEL(I),DEL(K)
   20 CONTINUE
      WRITE(6,32)DLAM,DK,RHO
      WRITE(6,5)
      DO 16 I=1,10
      K=I-1
      M=I+10
      WRITE(6,15)K,DEL(I),K,DEL(M)
   16 CONTINUE
      WRITE(6,31)
C
```

```
C       COMMENCE THE COMPUTATION BY ESTABLISHING VARIOUS FIXED
C       PARAMETERS.
C
        B=DLAM*RHO**2/4.0
        BH=SQRT(B)
        PI=3.141593
        C=1.0/SQRT(PI*DLAM)
        D=-3.0/2.0
        X=-0.9
        XINT=0.1
C
C       STATION CONTROL LOOP
C
        DO 100 I=1,20
        WFSQ=DBLE(B/(X+1.0))
        CALL SSUM(D,WFSQ,WSUM)
        AA=SQRT(X+1.0)
        SUM=SNGL(WSUM)
        BB=2.0*AA+BH*2.0*SUM
        TTERM=0.0
C
C       J LOOP
C
        DO 200 J=1,10
C
C       K LOOP
C
        DO 300 K=1,J
        ISIG=J-1
        IR=K-1
        IX=ISIG-IR
        CC=FLOAT(IFAC(ISIG))/(FLOAT(IFAC(IR))*FLOAT(IFAC(ISIG-IR)))
        CC=CC*(-1.0)**IR*X**IX*2.0
        DD=(X+1.0)**IR*AA/(2.0*FLOAT(IR)+1.0)
        A=-FLOAT(IR)+D
        CALL SSUM(A,WFSQ,WSUM)
        SUM=SNGL(WSUM)
        EE=B**IR*BH*SUM
        TERM=CC*(DEL(J)*DD-DEL(J+10)*EE)
        TTERM=TTERM+TERM
  300   CONTINUE
  200   CONTINUE
        POT=C*(BB-TTERM)
        WRITE(6,400)X,POT
  100   X=X+XINT
        WRITE(6,1000)
        STOP
        END
```

```
      SUBROUTINE SSUM(A,FSQ,SUM)
C
C     *****************************************************************
C
C                          SUBROUTINE SSUM
C
C     PRODUCE THE SOLUTION OF THE INTEGRAL 0.5*W**A*EXP(-W) ON THE
C     LIMITS FSQ=PI*DLAM/X+1 TO INFINITY.
C
C     *****************************************************************
C
      IMPLICIT REAL*4(A-C),REAL*8(D-H,O-V)
      SUM=0.0D0
C
C     OBTAIN SIGNIFICANT INTEGER POWER OF W WITH SIGN
C
      I=INT(A)
      F=DSQRT(FSQ)
      V=DSQRT(3.141592653589793)
      DEN=DEXP(FSQ)
C
C     TEST SIGN OF A
C
      IF(A)10,10,20
C
C     ''A'' NEGATIVE
C
   10 IF(I.EQ.0) GO TO 12
      GO TO 13
   12 TER1=-1.0D0
      GO TO 14
   13 DD=1.0D0
      EE=2.0D0
      I=-I
      I2=2*I-1
      DO 11 K=1,I
      I4=I2-2*(K-1)
      DD=DD*DFLOAT(I4)
      TER1=DFLOAT((-1)**(K+1))*EE/DD
      TER=TER1/(F**I4)
      EE=EE*2.0D0
      SUM=SUM+TER
   11 CONTINUE
   15 SUM=SUM/DEN
   14 SUM=(SUM-TER1*V*DERFC(F))/2.0D0
      GO TO 30
```

```
C
C     ''A'' POSITIVE
C
   20 IF(I.EQ.0) GO TO 22
      GO TO 23
   22 TER2=0.5D0
      SUM=F/DEN
      GO TO 24
   23 DD=1.0D0
      EE=1.0D0
      TER2=1.0D0
      I=I+1
      I2=2*I+1
      DO 21 K=1,I
      I3=I2-2*K
      TER1=F**I3
      TER=TER1*TER2
      DD=DD*DFLOAT(I3)
      EE=EE*2.0D0
      TER2=DD/EE
      SUM=SUM+TER
   21 CONTINUE
   25 SUM=SUM/DEN
   24 SUM=(SUM+TER2*V*DERFC(F))/2.0D0
   30 RETURN
      END
```

B-II. SONIC WIND TUNNEL WALL INTERFERENCE TECHNIQUE

Development of the Sandeman transonic wind tunnel wall interference solution for the steady case resulted in the following perturbation potential function from Section III-E:

$$\psi(X,0) = -\frac{1}{\sqrt{\pi\lambda}}\left\{\int_{-1}^{X}\frac{v(S)}{\sqrt{X-S}}\,dS\right.$$

$$\left.+\,2\sum_{N=1}^{\infty}\int_{-1}^{X}\frac{v(S)}{\sqrt{X-S}}\,\mathrm{EXP}\left[-\lambda\frac{N^2d^2}{X-S}\right]dS\right\}. \qquad \text{(III-60)}$$

To compare this solution with the cascade problem, (d) is related to blade spacing R by d = R/2 [Refer to Fig. 7]. Further, as shown in Eqn. (III-18),

$$v(S) = \psi_y(X,Y=0) = -1, \quad \text{for } K = 0.$$

It follows that the potential may be written:

$$\psi(X,0) = \frac{1}{\sqrt{\pi\lambda}}\left\{\int_{-1}^{X}\frac{1}{\sqrt{X-S}}\,dS\right.$$

$$\left.+\,2\sum_{N=1}^{\infty}\int_{-1}^{X}\frac{1}{\sqrt{X-S}}\,\mathrm{EXP}\left[-\frac{\lambda}{4}\frac{N^2R^2}{X-S}\right]dS\right\}. \qquad \text{(B-II-1)}$$

Integrating the first term and making the proper substitution in the second term one may write

$$\psi(X,0) = \frac{1}{\sqrt{\pi\lambda}}\left\{2\sqrt{X+1} + 2\sum_{N=1}^{\infty}\sqrt{\frac{\lambda N^2R^2}{4}}\int_{\frac{\lambda N^2R^2}{4(X+1)}}^{\infty}\frac{e^{-U}}{U^{3/2}}\,dU\right\}. \qquad \text{(B-II-2)}$$

The basic integral within the summation may be expressed in terms of the error function, i.e.,

$$\int_{F^2}^{\infty} \frac{e^{-U}}{U^{3/2}}\, dU = \frac{2\, e^{-F^2}}{F} - 2\sqrt{\pi}\, \mathrm{ERFC}(F) \qquad \text{(B-II-3)}$$

where

$$F^2 = \frac{\lambda N^2 R^2}{4(X+1)} \qquad \text{(B-II-4)}$$

and

$$\mathrm{ERFC}(F) = 1 - \mathrm{erf}(F) \; . \qquad \text{(B-II-5)}$$

Hence, the potential function:

$$\psi(X,0) = \frac{2}{\sqrt{\pi\lambda}} \left\{ \sqrt{X+1} + \sum_{N=1}^{\infty} \left(2\sqrt{X+1}\; e^{-\frac{\lambda N^2 R^2}{4(X+1)}} - \sqrt{\pi\lambda N^2 R^2}\;\; \mathrm{ERFC}\left[\sqrt{\frac{\lambda N^2 R^2}{4(X+1)}}\right]\right)\right\} . \qquad \text{(B-II-6)}$$

Computation of this function is easily carried out by computer. The series is truncated when the inclusion of additional terms only changes the sum by a value less than the convergence criterion, EPS.

Step Analysis

(Refer to the routine at the end of this appendix.)

The inputs to the routine are DLAM = λ and RHO = R. DK = K is inputed for accouting purposes in the output only. The output includes the above parameters and the value of

the perturbation velocity potential at each of twenty stations along the blade.

The control parameters and various fixed parameters are calculated prior to entering the control loops. DO LOOP 100 provides control to calculate the potential at each of the stations beginning at X = -0.9 and increasing by increments of $\Delta X = 0.2$. DO LOOP 50 computes the series summing terms until convergence is reached. Again, the function subroutine ERFC is a standard on line library routine performing the operation in (B-II-5).

Once the series is computed, statement 51 forms the potential on POT which is subsequently printed along with the applicable station location.

```
C
C     ****************************************************************
C
C     SOLUTION OF THE POTENTIAL ALONG THE BLADE BY THE  WALL
C     INTERFERENCE TECHNIQUE
C
C     ****************************************************************
C
   10 FORMAT('1'//////////' ',20X,'VALUE OF THE POTENTIAL ALONG THE BLAD
     1E--WALL INTERFERENCE TECHNIQUE'//'0',20X,'SOLUTION FOR THE FOLLOWI
     1NG PARAMETERS'/'0',20X,'DL=',F7.3,10X,'DK=',F7.3,10X,'RHO=',F7.3//
     1'0',20X,'STATION',13X,'POTENTIAL')
   52 FORMAT('0',20X,'X=',F5.2,10X,'POT=',1PE15.5)
 1000 FORMAT('1')
C
C     SET THE CONTROL PARAMETERS
C
C     ****************************************************************
C
      DLAM=0.2
      DK=0.0
      RHO=1.0
      WRITE(6,10)DLAM,DK,RHO
C
C     ****************************************************************
      PI=3.141593
      C=SQRT(PI*DLAM)
      X=-0.9
C
C     STATION CONTROL LOOP
C
      DO 100 I=1,20
      SUM=0.0
      A=X+1.0
      AH=SQRT(A)
C
C     SERIES CONTROL LOOP
C
      DO 50 N=1,500
      B=DLAM*N*N*RHO*RHO/(4.0*A)
      BH=SQRT(B)
      D=2.0*AH/EXP(B)
      TERM=D-N*RHO*C*ERFC(BH)
      TEMP=SUM
      SUM=SUM+TERM
C
C
C
```

```
C
C     TEST FOR CONVERGENCE
C
      IF(ABS(SUM-TEMP).LE.1.0E-5) GO TO 51
   50 CONTINUE
   51 POT=2.0*(AH+SJM)/C
      WRITE(6,52)X,POT
      X=X+0.1
  100 CONTINUE
      WRITE(6,1000)
      STOP
      END
```

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ABSTRACT

This thesis presents an analysis of transonic potential flow through an oscillating unstaggered thin plate cascade. A collocation technique is used involving the superposition of adjacent blade "isolated foil" potentials with interference potentials of unknown strength. Imposition of flow tangency requirements leads to integral equations for the unknown source distributions of the interference potentials. Results presented include the interference coefficients for the steady and oscillating cases.

The steady case is extended to the determination of the potentials along the blade and compares favorably with a parallel solution using a Laplace transform approach to the sonic wind tunnel wall interference problem.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
unsteady transonic cascade flow						
unsteady transonic						
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